Equitable Solutions in Game Representations and the Shapley Value

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Abstract

We show that *any* transferable utility game can be represented by an assignment of costly facilities to players, in which it is intuitively obvious how to allocate the total cost of all facilities amongst the players in an equitable manner. The equitable solution in the representation turns out to be the Shapley value of the game, and thus serves as an alternative justification of the value. We show that this approach extends also to the case when not all coalitions can form, provided those that can constitute a semi-algebra of sets (i.e., contain the grand coalition, and are closed under complements).

Key Words: TU game, characteristic function, Shapley value, assignment, representation, curtailed coalitions.

JEL Classification: C71, C72, D61, D63, D70, D79

1 Introduction

We show that *any* characteristic function v on player set N (also known as a transferable-utility, or TU, game on N) can be "represented" by an assignment of

(i) facilities that each player needs to use;

(ii) cost/benefit of each facility.

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Moreover, in this representation, all those who use any given facility are identical there[1] (in that the same facility suffices for an arbitrary set of its users, with no alteration in the cost/benefit generated collectively to them).

For ease of exposition, we shall from now on refer only to costs, with the explicit understanding that when a cost $\gamma < 0$ (or, $\gamma > 0$) it corresponds to a cost (or, benefit) in the colloquial sense.

Then the problem arises: how "should" the total cost, of all facilities, be allocated amongst N ?

An equitable solution comes almost unbidden to mind: at any facility, allocate its cost equally among all its (identical) users.

It turns out that this solution coincides with the Shapley value of v and thus serves to define the value on the domain of *all* characteristic functions.[2]

Worthy of note is the fact that the equitable solution emerges *just* from the assignment, without any consideration of coalitions in N, leave aside a characteristic function on N.

One may (if of a scholastic turn of mind) explicate two principles behind the equitable solution, both so compelling as to require little justification. The first embodies a form of *decentralization.* Note that the facilities themselves are decentralized, to begin with, in that the cost incurred by users at any facility does not depend on what is happening at other facilities (i.e., who goes to them, what their costs are, even how many of them there are, etc.; it is "independent of irrelevant information" regarding such externalities). It stands to reason that the cost allocaton should also inherit this property: it should be computed at each facility independently of the data regarding the others. This implies in particular that those who do not use a facility may simply be ignored there and not have to bear any of its cost; and is an aspect of what we have termed "equitable".

The second principle embodies the notion of *non-discrimination,* and makes up the other aspect of "equitable": at any facility, users who are identical must be treated in the same manner, i.e., allocated the same costs.

Finally let us point out that the game representation approach, outlined above, extends to the case when several coalitions are deemed infeasible, provided those that are feasible constitute a "semi-algebra" of sets (i.e., they contain N and are closed

[1] Since we are considering a representation, the more special its features, the better. However, our assumption that users are identical *vis-a-vis* any given facility, is without loss of generality. As was already shown in [2], and as will shortly become evident in our more general context, any scenario with non-identical users can be replicated with identical users by expanding the number of facilities.

[2] It also serves to describe the Shapley value to those who are not of a mathematical turn of mind.

under complements in N).

2 Assignments

Let N denote a finite set of *users* and K a finite set of *facilities*. Each user $n \in N$ is assigned a non-empty set $\psi(n) \subseteq K$ of facilities that he must use[3]; and each facility $k \in K$ has a non-empty set $\psi^{-1}(k) = \{n \in N : k \in \psi(n)\} \subset N$ of users assigned to it[4]. Furthermore there is a *cost* $\gamma(k) \in \mathcal{R}$ of facility k that accrues collectively to the users of k. (Here $\mathcal{R}$ denotes the reals; and the fact that $\gamma(k)$ depends only on k, and not on the set of users of k, implies in particular that these users are identical at k.)

The pair (ψ, γ) will be called an *assignment*[5]. In what follows, N will be fixed, while ψ, γ are allowed to vary in order to generate different assignments. (One may view (ψ, γ) as a bipartite graph, with disjoint sets N, K; an edge (n, k) iff $k \in \psi(n)$; and cost $\gamma(k)$ at each node $k \in K$.)

Given the assignment (ψ, γ), the natural question arises: how "should" the total cost $\sum_{k \in K} \gamma(k)$ be allocated to N ? The *equitable solution,* as was said, stipulates that at any facility k, its cost $\gamma(k)$ be split equally among all its identical users. Indeed, this solution was invariably given as being obvious, by a random assortment of lay people that the author posed the question to — high school students, fashion designers, actors, businessmen, politicians, *et al.* — all of them perfectly innocent of game theory. Yet their solution was the "Shapley value" of the "characteristic function" induced by (ψ, γ).

3 Characteristic Functions

Interpreting N to be a set of "players", let $\mathcal{N} = \{T : T \subseteq N\}$ denote the collection of all coalitions in N, and let $\mathcal{E} = \mathcal{E}(\mathcal{N})$ denote the Euclidean space whose axes are indexed by the elements of $\mathcal{N}$. A *characteristic function* on $\mathcal{N}$ is vector $v \in \mathcal{E}$ (equivalently, a function $v : \mathcal{N} \longrightarrow \mathcal{R}$), with $v(\emptyset) = 0$. The interpretation is that the component $v(T)$ represents the payoff that coalition T can guarantee to its members.

[3]The operative word is "must": not only is player n required to use all the facilities in $\psi(n)$, he is also debarred from using those in $K \diagdown \psi(n)$. In our model, facilities for n are exogenously thrust upon him, and he has no choice in the matter. (See also Remark 10.)

[4]If $\psi(n)$ (or, $\psi^{-1}(k)$) were empty, then n (or, k) could be dropped from the picture.

[5]This implicitly entails also varying K (which serves as the range of the correspondence ψ, and the domain of the map γ).

If v satisfies the condition $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \emptyset$, it is said to be *superadditive.* Our analysis holds for every v, superadditive or not, though it is conventional to assume superadditivity.

We shall show below that *any* characteristic function v on $\mathcal{N}$ is "induced" by an assignment (ψ, γ) of the kind we have described; in our parlance, v is "*represented*" by the assignment (ψ, γ). Assuming this for now, let us invoke the long forgotten "Axiom I" from Shapley's 1951 working paper [4] (where "value" refers to the allocation of $v(N)$ among the players). It is important to first stress that the domain, on which the value is sought to be defined in [4], does *not* consist of characteristic functions, but of what we call "scenarios" below (and Shapley called[6] "games" in [4]).[7]

Axiom I: The (Shapley) value (in any scenario) depends only on the characteristic function v (that the scenario induces).

In light of this axiom, and our result that *all* characteristic functions on $\mathcal{N}$ can be represented by assignments, the equitable solution determines the Shapley value on all scenarios. See Section 11 for more details.

Axiom I seems to have sunk into the subconscious of the subject, and indeed the value is routinely taken to depend on v *by definition* (as is the case even in a subsequent version [5] of [4]). However the axiom does have significant content, which is what presumably had led Shapley to state it in the first place. It calls attention to the fact that, in different scenarios $\Gamma^1, \Gamma^2, \ldots$, the value does not depend on the finer economic or engineering structure of the scenario, but only on its emergent "financial structure", i.e., on its characteristic function v. (In other words, for purposes of cost allocation, what matters are the costs incurred by coalitions, never mind *how* they incurred it.) This opens up the possibility of *reversing the gaze:* rather than go from a scenario Γ to the characteristic function v that Γ induces, and then figure out the value to players in v, one could instead start with v and conjure up a special scenario Γ^* which represents v and in which the value is intuitively obvious.

[6] However, since characteristic functions have subsequently come to be commonly called "transferable utility (TU) games", we use the term "scenarios" to prevent confusion.

[7] Scenarios constitute "representations" of the characteristic function they induce, even though they remained in the wings in [4] and only the characteristic function needed to be brought onto the stage.

4 The Characteristic Function Induced by an Assignment

Let (ψ, γ) be an assignment, with underlying sets N and K of users and facilities, as in Section 2. For any coalition $S \subset N$, let $\psi(S) = \cup_{n \in S} \psi(n)$ denote the set of facilities assigned to users in S, and let $v(S)$ be the cost of $\psi(S)$, i.e„

$$v(S) = \sum_{k \in \psi(S)} \gamma(k)$$

where summation over the empty set is understood to be zero. This defines the *characteristic function (or, TU-game)* v induced by (ψ, γ).

5 The Shapley Value

Recall that the Shapley value of the characteristic function v on $\mathcal{N}$ is an N-vector $\varphi(v) = (\varphi_n(v))_{n \in N}$ where:

$$\varphi_n(v) = \sum_{S \subset N \setminus \{n\}} \frac{|S|!(|N| - 1 - |S|)!}{|N|!} [v(S \cup \{n\}) - v(S)]$$

for $n \in N$. (Here $|X|$ denotes the cardinality of the set X , and 0!is understood to be1). As shown in ([4], [5]), φ is the unique solution (on the vector space of all characteristic functions on $\mathcal{N}$) that satisfies certain intuitively appealing axioms named "dummy", "symmetry","additivity" and "efficiency", the last tantamount to $\sum_{n \in N} \varphi_n(v) = v(N)$, i.e., $\varphi(v)$ is an allocation of the total proceeds $v(N)$ in the game.

6 The Representation of a Characteristic Function by an Assignment

Definition 1 *We say that the assignment* (ψ, γ) *is a* representation *of the characteristic function* v *if* v *is induced by* (ψ, γ).

Definition 2 *The* equitable solution *to* (ψ, γ) *is given by* $\tau(\psi, \gamma) = (\tau_n(\psi, \gamma))_{n \in N}$ *where*

$$\tau_n(\psi, \gamma) = \sum_{k \in \psi(n)} \frac{\gamma(k)}{|\psi^{-1}(k)|}$$

(Recall that $\psi^{-1}(k) = \{n \in N : k \in \psi(n)\}$ is nonempty by assumption.)

Theorem 3 *Let v be a characteristic function on $\mathcal{N}$. There exists a representation of v. Moreover, all representations of v have the same equitable solution, which coincides with the Shapley value of v.*

7 Characteristic Functions with Curtailed Coalitions

It might be useful to consider the case when not all coalitions can form. To this end, we make some definitions. Throughout, $\mathcal{C} \subset \mathcal{N}$ is a fixed collection of *feasible coalitions* and $\mathcal{E}(\mathcal{C})$ denotes the Euclidean space whose axes are indexed by the elements of $\mathcal{C}$.

Definition 4 *$\mathcal{C}$ is a* semi-algebra[8] if *(i) $N \in \mathcal{C}$; and (ii) $S \in \mathcal{C} \Longrightarrow N \diagdown S \in \mathcal{C}$.*

Definition 5 *A* characteristic function *v on $\mathcal{C}$ is vector in $\mathcal{E}(\mathcal{C})$ (equivalently, a function $v : \mathcal{C} \longrightarrow \mathcal{R}$), with $v(\emptyset) = 0$.*

Definition 6 *The assignment (ψ, γ) is a* representation *of the characteristic function v on $\mathcal{C}$ if v is induced by (ψ, γ) as before, i.e., both ψ and ψ^{-1} are nonempty-valued correspondences (from N to K, and K to N, respectively) and*

$$v(S) = \sum_{k \in \psi(S)} \gamma(k) \text{ for all } S \in \mathcal{C};$$

and if, furthermore,

$$\psi^{-1}(k) \in \mathcal{C} \text{ for all } k \in K.$$

To interpret the last condition, note that $\psi^{-1}(k)$ is precisely the coalition of players who *share* facility k in the assignment (ψ, γ). If this coalition were not feasible (i.e., $\psi^{-1}(k) \notin \mathcal{C}$), then (ψ, γ) would be inconsistent with the domain $\mathcal{C}$ of v, and hardly be entitled to "represent" v.

The following is a general version of Theorem 3.

[8] We use this term as an offshoot of the standard definition that $\mathcal{C}$ is an *algebra* if, in addition to (i) and (ii), one also has: $S, T \in \mathcal{C} \Longrightarrow S \cup T \in \mathcal{C}$.

Theorem 7 *Let $\mathcal{C}$ be a semi-algebra of coaltions in N and let v be a characteristic function on $\mathcal{C}$. There exists a unique representation (ψ^*, γ^*) of v with the minimum number of facilities; and all other representatins of v arise from (ψ^*, γ^*) by replicating facilities and splitting the cost of each facility among its replicas (while keeping unchanged the set of users across its replicas). It follows that the equitable solution is invariant of the representation.*

8 Proofs

8.1 Proof of Theorem7

Proof. Denote by $\mathcal{V} = \mathcal{V}(\mathcal{C})$ the $(|\mathcal{C}| - 1)$-dimensional subspace of $\mathcal{E}(\mathcal{C})$ consisting of all characteristic functions on $\mathcal{C}$ and denote $\mathcal{C}^* = \{S \in \mathcal{C} : S \neq N\}$ (and note: $\emptyset \in \mathcal{C}^*$). For any $S \in \mathcal{C}^*$ define $w_S \in \mathcal{V}$ by

$$w_S(T) = 0 \text{ if } T \subseteqq S \text{ , and } w_S(T) = 1 \text{ otherwise}$$

for all $T \in \mathcal{C}$. We claim that the set $\mathcal{W} = \{w_S : S \in \mathcal{C}^*\}$ constitutes a basis for $\mathcal{V}$. To see this, first define $u_S \in \mathcal{V}$, for any $\emptyset \neq S \in \mathcal{C}$, by

$$u_S(T) = 1 \text{ if } T = S, \text{ and } u_S(T) = 0 \text{ otherwise}$$

for all $T \in \mathcal{C}$, i.e., $\{u_S : \emptyset \neq S \in \mathcal{C}\}$ is the standard basis of $\mathcal{V}$ consisting of the "unit" vectors. It suffices to show that each $u_S \in$ Span $\mathcal{W}$. Let $k(1) < \ldots < k(T)$ be a list of the cardinalities of *all* the *nonempty* coalitions in $\mathcal{C}$. First take any $S \in \mathcal{C}$ with cardinality $k(1)$. It is clear that

$$u_S = w_\emptyset - w_S$$

and hence $u_S \in$ Span $\mathcal{W}$. Next assume inductively that $u_T \in$ Span $\mathcal{W}$ for any $T \in \mathcal{C}$ with cardinality $k(1), \ldots, k(t)$; and consider any $S \in \mathcal{C}$ with cardinality $k(t+1)$. Let $S_1, \ldots, S_l$ denote *all* the *strict* subsets of S that are in $\mathcal{C}$. Consider

$$w^* = w_\emptyset - w_S$$

In w^* observe that S and its strict nonempty subsets (that are in $\mathcal{C}$) get 1, and all other coalitions in $\mathcal{C}$ get 0. Therefore

$$u_S = w^* - u_{S_1} - \ldots - u_{S_l}$$

Now $w^* \in \text{Span}\,\mathcal{W}$ by the second-last display, and $u_{S_j} \in \text{Span}\,\mathcal{W}$ for $j = 1, \ldots, l$ by the inductive asssumption (since the cardinality of each S_j is $k(t)$ or less). Hence the last display implies that $u_S \in \text{Span}\,\mathcal{W}$. This establishes the claim.

Now consider any $v \in \mathcal{V}$. By the claim, there are unique scalars c_S such that

$$v = \sum_{S \in \mathcal{C}^*} c_S w_S$$

Let the set of facilities K^* correspond to (i.e., be indexed by) $\mathcal{C}^*$ and define the assignment (ψ^*, γ^*) as follows[9]:

$$\psi^*(n) = \{S \in \mathcal{C}^* : n \notin S\}, \text{ for } n \in N;$$

and

$$\gamma^*(S) = c_S, \text{ for } S \in \mathcal{C}^*.$$

It is readily verified that (ψ^*, γ^*) represents v. Indeed observe that for the facility k corresponding to $S \in \mathcal{C}^*$, our construction implies $\psi^{*-1}(k) = N \diagdown S \in \mathcal{C}$ (where the inclusion comes from the fact that $\mathcal{C}$ is a semi-algebra). Next note that, by the linear expression for v (third last display), we have

$$v(T) = \sum_{S \in \mathcal{C}^*} c_S w_S(T)$$

for any $T \in \mathcal{C}$. But $w_S(T) = 1$ if, and only if, $T \nsubseteqq S$; therefore

$$v(T) = \sum_{S \in \mathcal{C}^*, T \nsubseteqq S} c_S$$

Since the users in T are linked by ψ to precisely those facilities in $S \in \mathcal{C}^*$ such that $T \nsubseteqq S$ (see the fourth last display), i.e., $\psi^*(T) = \{S \in \mathcal{C}^* : T \nsubseteqq S\}$, it follows that (ψ^*, γ^*) represents v.

Finally, let $(\psi_\#, \gamma_\#)$ be an arbitrary representation of v with the set $K_\#$ of facilities. (Recall that the underlying set N of users is held fixed throughout.) At each $k \in K_\#$ define a characteristic function $v^k \in \mathcal{V}$ by

$$v^k(T) = \gamma_\#(k) \text{ if } T \cap \psi_\#^{-1}(k) \neq \emptyset$$

for all $T \in \mathcal{C}$. A moment's reflection reveals that

$$v = \sum_{k \in K_\#} v^k;$$

[9] If $c_S = 0$, one could of course drop facility S.

and that

$$v^k = \gamma_{\#}(k) w_{S(k)} \text{ , where } S(k) = N \diagdown \psi_{\#}^{-1}(k)$$

(Note $S(k) \in \mathcal{C}^*$ since, by assumption, $\emptyset \neq \psi_{\#}^{-1}(k) \in \mathcal{C}$ and $\mathcal{C}$ is a semi-algebra). This implies

$$v = \sum_{k \in K_{\#}} \gamma_{\#}(k) w_{S(k)}$$

From the fact that the linear expansion $v = \sum_{S \in \mathcal{C}^*} c_S w_S$ is unique (in terms of the basis $\{w_S : S \in \mathcal{C}^*\}$ of $\mathcal{V}$), we conclude that there is a partition $\{K_{\#}(S) : S \in \mathcal{C}^*\}$ of $K_{\#}$ into equivalence classes of the relation $\sim$ on $K_{\#}$ defined by: $k \sim l$ iff $S(k) = S(l)$, and that

$$v = \sum_{S \in \mathcal{C}^*} \left(\sum_{k \in K_{\#}(S)} \gamma_{\#}(k) \right) w_S$$

with

$$c_S = \sum_{k \in K_{\#}(S)} \gamma_{\#}(k)$$

Thus the arbitrary representation $(\psi_{\#}, \gamma_{\#})$ of v is merely a "*splintering*" *of the unique minimal representation* (ψ^*, γ^*) of v that was constructed earlier in the proof (with facilities K^*, corresponding to $\mathcal{C}^*$) ; in other words, $(\psi_{\#}, \gamma_{\#})$ is obtained from (ψ^*, γ^*) by replicating facilities of K^* and by splitting the cost of each $k \in K^*$ among the replicas of k, while leaving unchanged the set of users $\psi^{*-1}(k)$ at all the replicas of k. (The number of replicas of $k \in K^*$, and the split of the cost of k among its replicas, can be arbitrary; but modulo this arbitrariness, all representations of v are the same.) From this fact, it is evident that the equitable solution is invariant across all representations of v. ■

8.2 Proof of Theorem3

Proof. Let $\varphi : \mathcal{V}(\mathcal{N}) \longrightarrow \mathcal{R}^N$ denote the Shapley value (where, recall, $\mathcal{V}(\mathcal{N})$ denotes the vector space of all characteristic functions on $\mathcal{N}$) and let $v = \sum_{k \in K_{\#}} v^k$ as in the proof of Theorem 7, with $K_{\#}$ denoting the set of facilities of any representation $(\psi_{\#}, \gamma_{\#})$ of $v \in \mathcal{V}(\mathcal{N})$. Then we have $\varphi(v) = \sum_{k \in K_{\#}} \varphi(v^k)$ by the additivity of φ. But, in the characteristic function v^k, all the players in $T_k = \{n \in N : k \notin \psi_{\#}(n)\}$ are dummies and all the players in $N \backslash T_k$ are symmetric[10]. Hence $\varphi(v^k)$ awards 0 to players in T_k, and divides the total $v^k(N) = \gamma_{\#}(k)$ equally among the players in

[10]In making this statement, we are using the fact that $\mathcal{C} = \mathcal{N}$. See Remark 9.

$N \backslash T_k$ (using the dummy, symmetry and efficiency properties of φ). This implies that $\varphi(v)$ is the equitable solution $\tau(\psi_{\#}, \gamma_{\#})$. ■

9 Remarks

(1) **(Minimal Facilities)** It is evident from the proof of Theorem 7 that, for generic[11] $v \in \mathcal{V}(\mathcal{C})$, the unique minimal representation of v needs $(|\mathcal{C}| - 1)$ facilities.

(2) (**Tightness of the Assumptions)** The following examples show the need to assume (a) closure of $\mathcal{C}$ under complements; and (b) $\mathcal{C}$-measurability of the map ψ, i.e., $\psi^{-1}(k) \in \mathcal{C}$ for all $k \in K$.

First, let $\mathcal{C} = \{\emptyset, S, N\}$ for some $\emptyset \neq S \subsetneqq N$. Then any assignment which is $\mathcal{C}$-measurable has (upto replication) two facilities, one assigned to players in S and the other to players in N, with associated costs (say) γ and μ. This induces the characteristic function w on $\mathcal{C}$, with $w(\emptyset) = 0$, $w(S) = w(N) = \gamma + \mu$. Let us replace the costs γ, μ at the two facilities by arbitrary γ^*, μ^* with $\gamma^* + \mu^* = \gamma + \mu$. This yields a continuum of representations (one for every pair (γ^*, μ^*)) of w, all of which have different equitable solutions, showing the necessity of condition (a) for Theorem 7.

Next consider any w on the semi-algebra $\{\emptyset, S, N \diagdown S, N\}$ and extend it arbitrarily to v on $\mathcal{N}$ (the collection of all coalitions of N). Any representation of v will also serve as a representation of w if condition (b) were to be dropped, but the equitable solution of v varies widely depending on the extension v that is chosen. This demonstrates the necessity of condition (b).

(3) (**Decentralized Facilities)** The users of any facility k in an assignment are identical at k *only from the local viewpoint of* k, in the case $\mathcal{C} \neq \mathcal{N}$. Were we to look at the situation through the lens of coalitions $\mathcal{C}$ rather than the lens of a facility k, then the users at k may be far from identical *even if we confine attention to the local game v^k on $\mathcal{C}$ that is induced at the facility k.* (For instance, consider the case when both i and j are users of k, but i belongs to only small-sized coalitions in $\mathcal{C}$ while j belongs to only large-sized coalitions.) Note that the distinction between the k-lens and the $\mathcal{C}$-lens comes to the fore in the new territory $\mathcal{C} \neq \mathcal{N}$ that we have explored in Section 7. As for the standard case $\mathcal{C} = \mathcal{N}$, all users of k are symmetric in v^k and all non-users are dummies, even when seen through the $\mathcal{N}$-lens introduced in [4], [5]; thus the view is the same from either lens. (Nevertheless, part of our message is that here too we may avoid the consideration of coalitions, since the simple-minded k-lens

[11] Precisely: for all $v \in \mathcal{V}(N)$, after deleting a fiinite number of lower-dimensional subspaces.

of the representation suffices to reveal what can be seen through the more complex $\mathcal{C}$-lens of the characteristic function[12].)

(4) (**Other representations?**) Given a characteristic function v on $\mathcal{N}$, is it possible to conjure up a representation of v in which the intuitive solution is different from the Shapley value of v? In this paper, a "representation" is *defined* to be an assignment, and then the answer is no. The question will become meaningful if a broader notion of "representations", and of their "intuitive solutions", can be made mathematical. The author has no clue of how to go about this.

10 A Narrative for the Assignment

We present here a narrative that elaborates on how a characteristic function v is induced by an assignment (ψ, γ) in accordance with our definition. For simplicity, let us focus on the case where the induced v is superadditive (this is an implied restriction on pairs (ψ, γ) .)

Assume first that each $n \in N$ *must* use facilities in $\psi(n)$ – no more and no less – otherwise a huge penalty (say, $-\infty$) is levied on n. Next assume that the only way for a user to obtain access to facilities is for him to be part of a coalition which registers itself with a central authority (coalitions can be singletons). Once a coalition S registers, the authority allocates to S an "industrial park", indexed by S, at which only the members of S are allowed to enter and at which precisely all the facilities in $\psi(S)$ are available (along with their attendant costs given by γ). Now a little reflection reveals that the best that S can *guarantee* to itself, is to go as a block to the authority, since there is no incentive for S to split into smaller subsets (on account of superadditivity), and since *one* of the many courses of actions (rational or not) available to those in $N \backslash S$ is to go off on their own. (By going as a block, S incurs the cost $v(S) = \sum_{k \in \psi(S)} \gamma(k)$; while by going off, $N \backslash S$ can ensure that S does no better.)

When v is not superadditive, one might need to strengthen the narrative, and further assume that once players enter any park S they can never leave it, or interact in any way with others outside of the park. (We skip the details.)

The point is not whether the above narrative is realistic or interesting. It may not even be happening on earth but on some other (to our prejudiced minds) wierd planet. All that matters is that the narrative be logically coherent, with no internal contradictions, and that the numbers $v(S)$ be well-defined for every $S \in \mathcal{C}$,

[12] Indeed, this local (we may even call it "parochial") viewpoint is at the heart of the decentralization, leading to the economy and simplicity of the cost allocation rule.

compatible with the interpretation

$v(S) =$ maximum payoff that S can guarantee itself, no matter what players in $N \backslash S$ do

We admit any such narrative in (ψ, γ) as a scenario in the domain to which Axiom 1 applies.

11 An Axiomatic Framework

Let $\mathfrak{S}$ be an abstractly given set of "scenarios" on player-set N. Here by a scenario we mean any interaction among the players, in which it is clear how to define the amounts $v(S)$ that coalitions $S \in \mathcal{C}$ can guarantee to their members (see Section 5). For our purposes, it is not necessary to be more specific.

Further, let $\mathfrak{A}$ denote the set of *all* assignment scenarios on N as defined earlier; and recall that $\tau : \mathfrak{A} \longrightarrow \mathcal{R}^N$ denotes the equitable solution on $\mathfrak{A}$.

Finally let $\theta : \mathfrak{S} \longrightarrow \mathcal{R}^N$ be a payoff map to the players as (implicitly) in [4].

Axiom II: $\mathfrak{A} \subset \mathfrak{S}$ and θ coincides with τ on $\mathfrak{A}$.

In other words, the domain of θ is rich enough to include every assignment, and on each assignment it prescribes the equitable solution. Our theorem may then be restated: *the unique map θ which satisfies Axioms I and II is the Shapley value.*

As was pointed out, Axiom II can be split into two parts, requiring θ to satisfy decentralization and non-discrimination. Also note that Axiom I is only needed if $\mathfrak{S}$ extends beyond $\mathfrak{A}$.[13]

12 Related Literature

By way of a concrete instance of the assignment (ψ, γ), think of N as a set of identical planes, and of $\psi(n)$ as the set of cities that plane n must fly to.[14] If $\gamma(k) < 0$, it represents the cost of building the runway at city k (a runway, once built, accomodates any number of planes since they are of identical make); and if $\gamma(k) > 0$, one may think of it as a "subsidy" given by city k (collectively to the planes that visit its remote location). (If $\gamma(k) = 0$, city k can be dropped from the picture.) These costs (benefits) may vary across cities on account of different costs of labor, land, location, etc.

[13] This section was prompted by a query from Ori Haimanko.

[14] The daily flight path of each plane is a loop which traverses the cities in $\psi(n)$ (in any order), taking off and landing at each city in $\psi(n)$ exactly once.

This picture is based on [3], which focused attention on a single airport used by different-sized planes; and on [2] which observed that the model of [3] can be recast with identical-sized planes that have different flight paths among multiple airports. However the analysis in [2] (as in [3]) was restricted to the case of costs $\gamma(k) < 0$, which severely limited the class of characteristic functions that could be generated. The fact was missed in [2] that, by letting $\gamma(k)$ take on arbitrary values in $\mathcal{R}$, one can generate all possible characteristic functions

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