On the optimal design of a Financial Stability Fund

Working Paper 2018-06

ARPAD ABRAHAM, EVA CARCELES-POVEDA, YAN LIU, RAMON MARIMON

January, 2018

Stony Brook University

On the optimal design of a Financial Stability Fund*

Árpád Ábrahám † Eva Carceles-Poveda‡ Yan Liu§ Ramon Marimon¶

January, 2018

Abstract

A Financial Stability Fund set by a union of sovereign countries can improve countries' ability to share risks, borrow and lend, with respect to the standard instrument used to smooth fluctuations: sovereign debt financing. Efficiency gains arise from the ability of the fund to offer long-term contingent financial contracts, subject to limited enforcement (LE) and moral hazard (MH) constraints. In contrast, standard sovereign debt contracts are uncontingent and subject to untimely debt roll-overs and default risk. We develop a model of the *Financial Stability Fund* (*Fund*) as a long-term partnership with LE and MH constraints. We quantitatively compare the constrained-efficient *Fund* economy with the incomplete markets economy with default. In particular, we characterize how (implicit) interest rates and asset holdings differ, as well as how both economies react differently to the same productivity and government expenditure shocks. In our economies, 'calibrated' to the euro area 'stressed countries', substantial efficiency gains are achieved by establishing a well-designed *Financial Stability Fund*; this is particularly true in times of crisis. Our theory provides a basis for the design of a *Fund* - for example, beyond the current scope of the *Euroepan Stability Mechanism* (ESM) - and a theoretical and quantitative framework to assess alternative risk-sharing (shock-absorbing) facilities, as well as proposals to deal with the euro area 'debt overhang problem'.

JEL classification: E43, E44, E47, E63, F34, F36

Key words: Recursive contracts, debt contracts, partnerships, limited enforcement, moral hazard, debt restructuring, debt overhang, sovereign funds.

*We thank participants in seminars and conferences where versions of this work have been presented for their comments; in particular, Marco Bassetto, James Costain, Alessandro Dovis, Aitor Erce, Andreja Lenarcic and Rody Manuelli. **This is not a final ADEMU WP version**.

†European University Institute

‡SUNY at Stony Brook

§Wuhan University

¶European University Institute, UPF - Barcelona GSE, CEPR and NBER

1 Introduction

> "For all economies to be permanently better off inside the euro area, they also need to be able to share the impact of shocks through risk-sharing within the EMU."

This quote from the *Five Presidents' Report* (2015) recognizes a widely accepted fact: without a Federal Budget, or an institutional framework with similar fiscal automatic stabilizers for the euro area, it is unlikely that it will efficiently exploit its capacity for risk-sharing local or country risks with only private risk sharing and the existing EMU institutions[1]. In the aftermath of the financial and euro crises, with the subsequent upsurge of social unrest and discontent, the Five Presidents' call seems timely and urgent[2].

We develop a dynamic model of a *Financial Stability Fund* (*Fund*) as a long-term partnership addressing three features that are usually seen as the most problematic for a risk-sharing institution to be sustainable. First, risk-sharing transfers should not become persistent, or permanent, transfers, beyond the level of redistribution that partners would accept at any point in time, i.e. *ex-post* not only *ex-ante*. Second, as in any insurance contract, the *Fund* must take into account *moral hazard* problems, e.g. by avoiding current political costs, governments may increase future social and economic risks, and it is not possible to make the *Financial Stability Fund* contract conditional on the effort a country makes to reduce future liabilities and risks[3]. Third, risk-sharing among *ex-ante* equal partners without debt liabilities is relatively easy to design and achieve but, for example, this is not the case among European countries – in particular, the euro crisis has left a 'debt overhang problem' that aggravates the euro area divide. Our *Financial Stability Fund* accounts for the first two features by taking them as constraints, and accounts for the third by having country-specific long-term contracts and accounts, which can adapt to the EU diversity and may be able to solve existing 'debt overhang problems'. In sum, the *Financial Stability Fund* is a *constrained efficient mechanism* that can enhance countries' ability to share risks, borrow and lend, with respect to the standard instrument used to smooth fluctuations: sovereign debt financing.

In this paper we develop a model of *Financial Stability Fund* contracts between a risk-averse, relatively small and impatient borrower (the sovereign country) and a risk-neutral lender (the fund itself), and we evaluate its quantitative performance by calibrating the model to 1980-2015 data from the euro area 'stressed countries', to see how they would have performed during this period had they been in the fund. However, to assess the efficiency of the *Fund* it must be in relation to some other risk-sharing mechanism. We use, as a benchmark, an incomplete markets model where sovereign countries issue long-term defaultable debt in order to smooth their consumption.

[1] For example, Furceri and Zdzienicka (2015) estimate that the percentage of non-smoothed GDP shocks is 20% in Germany, 25% in the United States, but 70% in the Euro Area (15) 1978 - 2010. Using their methodology, M. Lanati estimates that it is 83% non-smoothed for EA(19), 1995 - 2015. Beraja (2016) has performed the counterfactual exercise of having the United States being Independent States. He finds, using a ' Semi-Structural Methodology' that, if the employment rate's cross-state standard deviation was 2.6% in 2010, it would have been 3.5% had it not been a fiscal union.

[2] However, almost surprisingly, they leave the task for a future date:

> "In the medium term, as economic structures converge towards the best standards in Europe, public risk-sharing should be enhanced through a mechanism of fiscal stabilisation for the euro area as a whole."

[3] Although 'austerity programs', to gain financial assistance in the euro crisis, attempted this.

In order to properly compare the *Fund* economy with the incomplete markets economy with defaultable debt (IMD), we 'decentralize' the fund contract generating the appropriate prices. For example, both in the *incomplete markets* economy with default, and in the two-sided limited-enforcement *Fund* economy, interest rates may differ from the risk-free rate. In the former, the *positive spreads* reflect the risk of default, while in the latter the *negative spreads* reflect the risk that the lender's participation – or, with moral hazard, borrower's incentive compatible – constraints become binding. Lower interest rates deter the lender from lending, thus implementing the *Fund* lender's enforcement constraint, which in our simulations is a tight constraint: 'at any point and state the lender's expected profits must be non-negative'. In both regimes, default is costly, resulting in autarky, with only a small probability of being able to join the incomplete markets economy and issue long-term defaultable debt.

It is interesting to note how the *Fund* mechanism compares with – *de facto* defaultable – long-term uncontingent sovereign debt contracts, currently in place, when the risk-averse borrowing country is subject to similar shocks to those to which the euro area 'stressed countries' have been exposed, in the last ten years. Without debt crises, the real euro crisis would not have been so severe, nor would it have turned into a recession; consumption smoothing and, therefore, the welfare of the borrowing country would have been higher, even if ex-post permanent transfers from the risk-neutral fund were set to zero.

We are not the first to address these issues, and there are many proposals for how risks could be shared in a monetary union, as there are for dealing with sovereign debt overhang problems. For example, as an implicit criticism of different proposals to issue some form of joint-liability eurobonds, Tirole (2015) emphasises the asymmetry issue: the optimal (one-period) risk-sharing contract with two symmetric countries is a joint liability debt contract serving as a risk-sharing mechanism, while the optimal contract between two countries with very different distress probabilities is a debt contract with a cap and no joint liability, where the cap depends on the extent of solidarity, which is given by the externality cost of debt default on the lender. With long-term relationships – as they are among sovereign countries that form a union – better contracts can be implemented: the *Fund* contracts are *constrained efficient* and can be implemented as long-term bonds with state-contingent coupons.

On the more practical side, a positive development within the euro crisis was the creation of the *European Stability Mechanism* in 2012, which treaty (Ch. 4 Art. 12.1) establishes as its first principle that:

> If indispensable to safeguard the financial stability of the euro area as a whole and of its Member States, the ESM may provide stability support to an ESM Member subject to strict conditionality, appropriate to the financial assistance instrument chosen.

While this first principle assesses the need to have contingent contracts, it also limits its funding to *extreme* events. Conditionality is a property of the optimal long-term contract that we characterize, but in contrast with the *ESM*, the *Fund* is designed as a risk-sharing mechanism, not as a crisis-resolution mechanism, and its conditionality is based on *ex-post* realisations, not on *ex-ante* promises to reform, which often require *ex-post* renegotiations.

Our model of the *Fund* as a partnership builds on the literature on dynamic optimal contracts with enforcement constraints (e.g. Marcet and Marimon 2017), as well as on the related literature on price decentralization of optimal contracts (e.g. Alvarez and Jermann 2000, Krueger et al. 2008).

Our benchmark incomplete markets economy with long-term debt with default, builds on the model of Chatterjee and Eyigungor (2012), who extend the sovereign default models of Eaton and Gersovitz (1981) and Arellano (2008) to long-term debt.

The paper is organized as follows. Section 2 presents the economy with incomplete markets and sovereign defaultable long-term debt. Section 3 develops the *Fund* mechanism and Section 4 shows how to decentralize the Fund contract with state-contingent long term bonds. Section 5 discusses the calibration and data sources. Section 6 quantitatively compares the IMD and Fund regimes *without moral hazard*, concluding with a welfare comparison and showing the ability of the Fund to confront 'debt overhang' problems. Section 7 extends the calibrated model to account for *moral hazard*, showing how allocations and bond prices change when *incentive compatibility* constraints are introduced. Section 8 shows how a simpler – less state contingent – Fund contract can be designed and decentralized. Section 9 concludes.

2 The economy and the benchmark case of sovereign debt financing

We consider a standard infinite-horizon representative agent economy, where the agent has preferences for current leisure, $l = 1 - n$, consumption, c, and effort, e, represented by $U(c, n, e) := u(c) + h(1 - n) - v(e)$ and discounts the future at the rate β. We make standard assumptions on preferences[4]. The agent has access to a decreasing returns labor technology $y = \theta f(n)$, where $f' > 0$, $f'' < 0$ and θ is a productivity shock, assumed to be Markovian; $\theta \in (\theta_1, ..., \theta_N)$, $\theta_i < \theta_{i+1}$. The economy is a small open economy in a world with no uncertainty with interest rate r satisfying $1/(1 + r) \geq \beta$; an inequality that, in general, we will assume to be strict. In order to borrow and save, the agent, which we also identify with the government of the country, may have access to different financial technologies, which will define different regimes, which we also call different economies.

The country also faces government expenditure shocks $G = G^c + G^d$, which together with the productivity shock defines the exogenous state, denoted by $s = (\theta, G)$. G^c takes discrete values from $G^c \in \{G^c_1, \dots, G^c_{N_G}\}$ and is a Markov process with transition probability $\pi^{G^c}(G'|s, e)$, and G^d is i.i.d. over time with continuous distribution ν over $G^d = \big[-\bar{m}, \bar{m}\big]$. In addition, G^c and G^d are independent with each other. The interpretation is that G^c are government expenditures and the distribution of next period expenditures depend on the policies that the government implements in the current period. In particular, the government can have a a better distribution of tomorrow's expenditures if it exercises sufficient effort in the current period (e.g. politically costly reforms are more likely to result in lower government expenditures). G^d is a residual shock that cannot be affected by government actions[5]. More precisely, we assume that given the current state, $s = (\theta, G)$, the next period realizations of θ and G are independent and only the latter depends on effort. That is

$$\pi(s'|s, e) = \pi^\theta(\theta'|\theta)\pi^G(G'|G, e)$$

[4]In particular, we assume that $(c, n, e) \in \mathbb{R}^3_+, n \leq 1$, and u, h, v are differentiable, with $u'(c) > 0, u''(c) < 0, h'(c) > 0, h''(c) < 0$ and $v'(c) > 0, v''(c) > 0$.

[5]The introduction of G^d is for technical reasons, as in Chatterjee and Eyigungor (2012). Notice that the composite G shock admits a Markov structure as well, with state space $G = \cup_i \big[G^c_i - \bar{m}, G^c_i + \bar{m}\big] \subset \mathbb{R}$ and transition kernel $\pi^G = \pi^{G_c} \otimes \nu$.

We assume that the cost of this effort are expressed in utility terms given by $v(e)$. We assume that high effort increases the probability of lower government expenditure, in this sense we can think about effort as 'austerity' measures with utility costs which reduce primary deficit. We assume that both $v(\cdot)$ and $\pi^G(G'|s,e)$ are continuous and twice differentiable in effort, moreover we assume that $v(\cdot)$ is convex.

2.1 The incomplete market model with long-term bond financing

The incomplete market model is a quantitative version of the seminal model by Eaton and Gersovitz (1981). We integrate three modeling advances in the recent literature, namely endogenous labor and output, long-term bonds, and an asymmetric default penalty, to achieve a more complete description of the business cycle dynamics of an small open economy with sovereign debt. We detail the specification of the baseline incomplete market model in this section.

In the incomplete market model, the borrower can issue or purchase *long-term* bonds, which promise to pay constant cash flows across different states. We model the long-term bond in the same way as Chatterjee and Eyigungor (2012).

A unit of long-term bond is parameterized by (δ, κ), where δ is the probability of continuing to pay out the coupon in the current period, and κ is the coupon rate. Alternatively, $1-\delta$ is to the probability of maturing in the current period, and this event is independent over time. The size of each bond is infinitesimal and the payment of each bond is independent in cross-section. As a result, on average one unit of bond (δ, κ) will repay $(1-\delta)+\delta\kappa$ in the current period for sure. It also follows that the bond portfolio has a recursive structure, in which only the size of total outstanding debt b matters, regardless when a particular issue of the bond enters into the portfolio. Moreover, δ directly captures the duration of the bond: if $\delta = 0$ the bond becomes the standard one-period debt, and in general, the average maturity of the bond equals to $1/(1-\delta)$, which is increasing in δ. The coupon rate κ provides a flexible way to capture the coupon payment: $\delta\kappa$ equals to the coupon payment on each unit principal of outstanding debt.

For an outstanding bond portfolio of size b, its cash flow stream is given by $(1-\delta)b + \delta\kappa b$, $\delta(1-\delta)b + \delta^2\kappa b$, ... When there is no default, the price of a unit of a *riskless* long-term bond (δ, κ), given a constant discount rate r, is:

$$q = \sum_{t=0}^{\infty}[(1-\delta)+\delta\kappa]\frac{\delta^t}{(1+r)^{t+1}} = \frac{(1-\delta)+\delta\kappa}{r+1-\delta},$$

with the corresponding risk free yield to maturity:

$$r = \frac{(1-\delta)+\delta\kappa}{q} - (1-\delta)\,.$$

Alternatively, if we define $Q \equiv \frac{q}{1-\delta+\delta\kappa+\delta q}$, then $Q = \frac{1}{1+r}$.

2.2 The budget Constraint and default

Let b_t denote the size of the bond portfolio (δ, κ) held by the borrower at the beginning of time t. Following the convention in the literature, $b_t \geq 0$ means holding assets while $b_t < 0$ means having debt. The borrower first makes a decision on whether to default on the promised bond payment of the entire bond portfolio b_t.

No default When the borrower chooses not to default, then the bond payment $(1-\delta)b_t + \delta\kappa b_t$ will be settled as promised: if $b_t \geq 0$, then the bond payment is part of the borrower's time t income; else if $b_t < 0$, then the borrower will make the required payment to the lender. Choosing not to default allows the borrower to stay in the bond market, so that the borrower may choose the bond holding position b_{t+1} for the next period. The difference between b_{t+1} and the remaining principal δb_t is the net issuance at time t. Due to the recursive structure of the long-term bond, the cash flows starting from $t+1$ onward of both b_{t+1} and δb_t are proportional, and therefore the same unit bond price applies to both. As to be explained below, the bond price is a function of the exogenous shock s_t and the bond position b_{t+1} for the next period, thus we use $q(s_t, b_{t+1})$ to denote this function. It follows that when the borrower chooses not to default, the budget constraint is as follows:

$$c_t + q(s_t, b_{t+1})(b_{t+1} - \delta b_t) \leq \theta_t n_t^\alpha - G_t + (1 - \delta + \delta\kappa)b_t.$$

Default Upon choosing default, the borrower is excluded from the bond market immediately and enters into autarky. As a result, the time t consumption is given by

$$c_t = \theta^p\left(\theta_t\right) n_t^\alpha - G_t.$$

The exclusion lasts for a random number of periods. If the borrower is excluded from the market in the previous period, then with probability $\lambda < 1$ the borrower regains access to the bond market in the current period, and with remaining probability $1-\lambda > 0$ the borrower stays in autarky. Moreover, upon regaining access to the bond market, the borrower starts from a zero bond position.

Besides the exclusion from the bond market, the borrower also suffers from a productivity penalty in autarky. As in Arellano (2008), the penalty takes the following form:

$$\theta^p\left(\theta\right) = \begin{cases} \bar{\theta}, & \theta \geq \bar{\theta} \\ \theta, & \theta < \bar{\theta} \end{cases} \quad \text{with } \bar{\theta} = \psi \mathbb{E}\theta,$$

which is *asymmetric* in the sense that the magnitude of the penalty is zero for lower than average" productivity states, while it is equal to $\theta - \bar{\theta}$—increasing in θ—for higher than average productivity states. The level of the penalty is parameterized by $\psi > 0$. Given that $0 < \theta_1 < \cdots < \theta_{N_\theta}$, on the one hand the penalty becomes a benefit if $\psi \geq \theta_{N_\theta}/\mathbb{E}\theta$; and on the other hand, the penalty ceases to be effective if $\psi < \theta_1/\mathbb{E}\theta$, since the borrower can always choose to have zero debt while enjoying higher productivity levels. An asymmetric penalty is crucial for the quantitative performance of models with sovereign debt and default. When the penalty is properly specified, it creates incentives for the borrower to borrow more in good states while deterring default temptation by harsh punishment, and these high levels of debt then induce the borrower to choose default in bad states where the penalty is zero.

2.3 Recursive Formulation

If b the size of the long-term bond portfolio held by the borrower at the beginning of a period[6], then (s, b), $s = (\theta, G)$, is the state. Let $V_n^b(b,\, s)$ denote the value function of the borrower, in the incomplete

[6] We assume that $b \in \mathcal{B} = [b_{\min}, b_{\max}]$, with $-\infty < b_{\min} < 0 \leq b_{\max} < \infty$., where we will choose $b_{\min}$ and $b_{\max}$ so that in equilibrium the bounds are not binding.

market economy, at the beginning of a period before any decisions are made. The value function when the borrower chooses not to default satisfies

$$\begin{aligned} V_n^b(b,\, s) &= \max_{c,n,e,b'} \left\{U(c,n,e) + \beta \mathbb{E}\left[V^b(b',\, s') \mid s, e\right]\right\} \\ \text{s.t.} \quad & c + G + q(s,b')(b' - \delta b) \leq \theta f\,(n) + (1 - \delta + \delta\kappa) b, \end{aligned} \tag{1}$$

where, taking into account that default can occur next period,

$$V^b(b,\, s) = \max\{V_n^b(b,\, s), V^a(s)\}, \tag{2}$$

and $V^a(s)$ is the value upon default, given by

$$\begin{aligned} V^a(\, s) = \max_{n,e} & \{u\,(\theta^p(\theta) f\,(n) - G) + h(1-n) - v(e) \\ & + \beta \mathbb{E}\left[\left(1 - \lambda^i\right) V^a(s') + \lambda^i V^b\,(0,\, s') \mid s, e\right], \end{aligned} \tag{3}$$

where λ^i is the probability to come back to the market and be able to borrow again. We denote the choices when there is no default, given by (1), by $(c(b,\, s), n(b,\, s), e(b,\, s), b'(b,\, s)$ and those in autarky, given by (3), by $(n^a(s), e^a(s))$. Note that since we assume effort, e, is not observable or contractable, the lender should try to infer the effort choice based on all its current information – in particular, the state $(b,\, s)$ – but, as it will become clear, in fact, the effort does not depend on b and could also be denoted by $e(s)$ when there is no default. The bond price has also a recursive structure. Let the default decision be given by

$$D(\, s, b) = 1 \text{ if } V^{ai}(\, s) > V_n^{bi}(b,\, s) \text{ and } 0 \text{ otherwise;}$$

therefore, the expected default rate is $d(s,b') = \mathbb{E}\left[D(s',\, b') \mid \, s, e(s,b)\right]$ The equilibrium bond pricing function $q(s,b')$ satisfies the following recursive equation:

$$q(s,b') = \frac{\mathbb{E}\left[\left(1 - D(s',b')\right)\left[(1-\delta) + \delta\left[\kappa + q(s',\, b''(s',\, b'))\right]\ \right] \mid \, s, e(s,b)\right]}{1+r},$$

which can also be expressed as:

$$\begin{aligned} q(s,b') \quad &= \quad \frac{(1-\delta) + \delta\kappa}{1+r}\left(1 - d(s,b')\right) \\ &+ \quad \delta \frac{\mathbb{E}\left[\left(1 - D(s',\, b')\right) q(s',\, b''(s',\, b') \mid \, s, e(s,b)\right]}{1+r}, \end{aligned} \tag{4}$$

For the one-period bond ($\delta = 0$), this reduces to $q(s,b') = \frac{1-d(s,b')}{1+r}$. The implied interest rate (i.e. yield to maturity) of the long-term bond is given by

$$r^i(s,b') = \frac{(1-\delta) + \delta\kappa}{q(s,b')} - (1-\delta)\,,$$

resulting in a *positive spread* $r^i\,(s,b') - r \geq 0$, which is strictly positive if $d(s,b') > 0$.

In order to keep track of debt flows, it is useful to define the *primary surplus* – or *primary deficit* if negative – which is given by

$$q(s,b')(b' - \delta b) - (1 - \delta + \delta\kappa) b = \theta f\,(n) - (c+G)$$

2.4 The effort decision

The optimal policies $c^b(b,s)$, $n^b(b,s)$, $b'(b,s)$ and $n^a(s)$ are standard dynamic programming solutions to (1) and (3), respectively. The effort policy function when there is no default in state $s=(\theta,G)$, $e^b(b,s)$, is given by

$$v'(e^b(b,s)) = \beta \sum_{s'} \pi^\theta(\theta'|\theta) \frac{\partial \pi^G(G'|G,e^b(b,s))}{\partial e} V^b(b',s'),$$

where b' is the optimal choice of debt in (1). Similarly, the effort policy function when there is default in state $s=(\theta,G)$, $e^a(s)$, is given by

$$v'(e^a(s)) = \beta \sum_{s'} \pi^\theta(\theta'|\theta) \frac{\partial \pi^G(G'|G,e^a(s))}{\partial e} \left[(1-\lambda^i)V^a(s') + \lambda^i V^b(0,s')\right],$$

since in (3) the choice of debt is predetermined to be zero.

3 The ***Financial Stability Fund*** as a long-term contract

An economy with a *Financial Stability Fund* (Fund) is modeled as a long-term contract between a fund, or *Fund* (also called lender), who can freely borrow and lend in the international market, and an individual partner (also called country or borrower), who is 'the representative agent' of the small open economy. We assume that the manager cannot observe the effort of the partner – or, simply, that the effort is not contractable,– which implies that the long term contract will have to provide sufficient incentives for the country to implement a (constrained) efficient level of effort. In the *fund contract*, the country consumes c and the resulting transfer to the Fund manager is $\tau = \theta f(n) - (c+G)$; i.e. when $\tau < 0$ the country is effectively borrowing. We consider that there is two-sided limited enforcement; that is, both the Fund manager and the lender can renege of their contract and pursue their outside options at any time-state.

In state $s^t = (s_0, \ldots, s_t)$, the outside value for the borrower country is the value of being in the incomplete market economy after default – that is $V^a(s^t)$, given by (3). In other words, once a country has joined the fund if it ever quits, or does not fulfil the *Fund* contract, the country is not allowed back and goes into autarky and then with probability λ^i is able to borrow in the private market. The outside option of the lender is $Z \leq 0$, at any s^t, which is determined by the willingness of the Fund (the lender) to accept some level of redististribution or to avoid that the country breaks away[7]. Whether it is ex-post altruisitic or self-interested – as in Tirole (2015) – *solidarity*, Z is an important parameter when assessing the efficiency gains of establishing a Fund; in particular, if $Z=0$ the Fund may still be superior to other mechanisms, since it can still provide some level of risk-sharing and for the impatient borrower can always be a better 'borrowing mechanism'. In the next Section we show how Z constraints the paths of Fund transfers and its effect on prices.

With *two-sided limited enforcement*, denoted (2S), an optimal fund contract is a solution to the

[7] Our characterisation easily generalises to the case that the outside value of the manager (lender) is time-state dependent.

following problem:

$$\max_{\{c(s^t),n(s^t),e(s^t)\}} \mathrm{E}\left[\mu_{b,0}\sum_{t=0}^{\infty}\beta^t\left[U(c(s^t),\,n(s^t),e(s^t))\right]+\mu_{l,0}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t\tau(s^t)\mid s_0\right]$$

$$\text{s.t.}\qquad \mathrm{E}\left[\sum_{r=t}^{\infty}\beta^{r-t}\left[U(c(s^r),\,n(s^r),\,e(s^r))\right]\mid s^t\right]\geq V^a\left(s_t\right), \tag{5}$$

$$v'(e(s^t))=\beta\sum_{s^{t+1}|s^t}\pi^{\theta}(\theta'|\theta)\frac{\partial\pi(s^{t+1}|s_t,\,e(s^t))}{\partial e(s^t)}V^{bf}(s^{t+1}), \tag{6}$$

$$\mathrm{E}\left[\sum_{r=t}^{\infty}\left(\frac{1}{1+r}\right)^{r-t}\tau(s^r)\mid s^t\right]\geq Z, \tag{7}$$

$$\text{and}\qquad \tau(s^t)=\theta(s^t)f\left(n(s^t)\right)-c(s^t)-G(s^t),\ \forall s^t,t\geq 0,$$

where the first two constraints (5) and (7) are the intertemporal participation constraints for the borrower and the lender, respectively, and (μ_{b0},μ_{l0}) are initial Pareto weights. Here the notation is explicit about the fact that expectations are conditional on the implemented effort sequence as it affects the distribution of the shocks. The constraint (6) is the incentive compatibility constraint with respect to effort[8], where $V^{bf}(s^{t+1})$ is the value of the *Fund* contract to the borrower in state s^{t+1}.

By imposing equality in (6) we have implicitly assumed that effort is interior, that is $e>0$. The interpretation of this constraint is standard: the marginal cost of increasing effort has to be equal to the marginal benefit. The latter is measured as the change in life-time utility due to the change in the distribution of future shocks as a result of the increasing effort. Note that $V^{bf}(s^{t+1})$ can also be written explicitly as the continuation life-time utilities of the borrower for all continuation states from next period on. In particular, (6) can also be written as:

$$v'(e(s^t))=\beta\sum_{s^{t+1}|s^t}\pi^{\theta}(\theta'|\theta)\frac{\partial\pi(s^{t+1}|s_t,\,e(s^t))}{\partial e(s^t)}\mathbb{E}\left[\sum_{r=t+1}^{\infty}\beta^{r-(t+1)}\left[U(c(s^r),\,n(s^r),\,e(s^r))\right]\mid s^{t+1}\right].$$

It is known from Marcet and Marimon (2017) and Mele (2014) that we can rewrite the general

[8]Note that we have used the first-order condition approach here, that is, we have replaced by the agent's full optimization problem by its necessary first-order conditions of optimality. According to the results of Rogerson (1988), the first-order conditions are also sufficient if the $\pi^G(G'|s,e)$ functions satisfy the monotone likelihood ratio and the convex distribution function conditions described below.

MLR. The probability shifting function $\pi^G\left(G'\mid s,e\right)$ has the *monotone likelihood ratio* property if, for each $e\geq 0$ and s, the ratio $\frac{\partial\pi^G(G'|s,e)/\partial e}{\pi^G(G'|s,e)}$ is non-increasing in G'.

CDF. The functions $\pi^G(G'|s,e)$ satisfy the convex distribution function condition if $\frac{\partial^2 F_{\tilde{G}}(s,e)}{\partial e\partial e}$ is non-negative for every e, s and $\tilde{G}$ where $F_{\tilde{G}}(s,e)=\sum_{G'\leq\tilde{G}}\pi^G(G'|s,e)$.

fund contract problem as a saddle-point problem [9]:

$$\begin{aligned}
&\text{SP} \min_{\{\gamma_{b,t},\gamma_{l,t}\,\xi_t\}} \max_{\{c_t,n_t,e_t\}} \mathbb{E}\Bigg[\sum_{t=0}^{\infty}\beta^t \left(\mu_{b,t}U(c_t,\, n_t,\, e_t) - \xi_t v'(e_t)\right. \\
&\qquad\qquad \left. + \gamma_{b,t}\left[U(c_t,\, n_t,\, e_t) - V^a(s_t)\right]\right) \\
&\qquad + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \left(\mu_{l,t+1}\left[\theta_t f(n_t) - G_t - c_t\right] - \gamma_{l,t}Z\right) \mid s_0\Bigg] \\
&\mu_{b,t+1} = \mu_{b,t} + \gamma_{b,t} + \xi_t \frac{\partial\pi(s_{t+1}|s_t,e_t)/\partial e}{\pi(s_{t+1}|s_t,e_t)}, \quad \text{with } \mu_{b,0} \text{ given, and} \\
&\mu_{l,t+1} = \mu_{l,t} + \gamma_{l,t}, \quad \text{with } \mu_{l,0} \text{ given}
\end{aligned}$$

Here $\beta^t\pi(s^t|s_0)\gamma_b\left(s^t\right)$, $\beta^t\pi(s^t|s_0)\gamma_l\left(s^t\right)$ and $\beta^t\pi(s^t|s_0)\xi\left(s^t\right)$ are the Lagrange multipliers of the limited enforcement constraints (5), (7) and incentive compatibility constraint (6), respectively, in state s^t. That is, with one-sided limited commitment $\gamma_l\left(s^t\right) = 0$, $\forall t \geq 0$. Notice that, by construction, $\frac{\partial\pi(s_{t+1}|s_t,e_t)/\partial e}{\pi(s_{t+1}|s_t,e_t)} = \frac{\partial\pi^G(G_{t+1}|G_t,e_t)/\partial e}{\pi^G(G_{t+1}|G_t,e_t)}$; that is, with the incentive compatibility constraint (6), the co-state $\mu_{b,t+1}$ is a vector $\mu_{b,t+1}(G_{t+1} \mid G_t)$, while without (6) it is a number.

We will use a convenient **normalization**, in order to minimise the dimension of the co-state vector. Let $\eta \equiv \beta(1+r) \leq 1$ and normalize multipliers: $v_{i,t} = \gamma_{i,t}/\mu_{i,t}, i = b, l$, $\widetilde{\xi}_t = \frac{\xi_t}{\mu_{b,t}}$ and

$$\varphi_{t+1}(G_{t+1} \mid G_t, e_t) = \widetilde{\xi}_t \frac{\partial\pi(G_{t+1}|G_t,e_t)/\partial e_t}{\pi(G_{t+1}|G_t,e_t)}; \tag{8}$$

then, a new co-state vector is recursively defined as:

$$x_0 = \mu_{b,0}/\mu_{l,0} \text{ and } x_{t+1} = \frac{1 + v_{b,t} + \varphi_{t+1}}{1 + v_{l,t}}\eta x_t$$

With this normalization, $v_{b,t}$ and $v_{l,t}$ become the multipliers of the limited enforcement constraints, corresponding to (5) and (7), and φ_t the multiplier of the incentive compatibility constraint, corresponding to (6). With this normalization, the state and co-state vector is given by (x, s) and the saddle-point Bellman equation is given by

$$\begin{aligned}
FV(x,s) = &\text{SP} \min_{\left\{v_b,v_l\,,\widetilde{\xi}\right\}} \max_{\{c,n,e\}} \left\{x\left[(1+v_b)\,U(c,\, n,\, e) - v_b V^a(s) - \widetilde{\xi}v'(e)\right]\right. \\
&\left. + \left[(1+v_l)\left(\theta f(n) - G - c\right) - v_l Z\right] + \frac{1+v_l}{1+r}\mathbb{E}\left[FV(x',s') \mid s,e\right]\right\}
\end{aligned} \tag{9}$$

$$\text{where } x' = \frac{1 + v_b + \varphi(G' \mid G,e)}{1+v_l}\eta x \text{ and } \varphi(G' \mid G,e) = \widetilde{\xi}\frac{\partial\pi(G'|G,e)/\partial e}{\pi(G'|G,e)}.$$

Furthermore (see Marcet and Marimon (2017)), the *Fund* policy function takes the form:

$$FV\left(x,s\right) = xV^{bf}\left(x,s\right) + V^{lf}\left(x,s\right), \quad \text{with}$$

[9]Following Marcet and Marimon (2017), we only consider saddle-point solutions and their corresponding saddle-point multipliers; that is, given $F(a,\lambda)$, (a^*,λ^*) solves SP $\min_\lambda \max_a F(a,\lambda)$ if and only if $F(a,\lambda^*) \leq F(a^*,\lambda^*) \leq F(a^*,\lambda)$, for any feasible action a and Lagrangian multiplier λ.

$$V^{bf}(x,s) = U(c(x,s),\, n(x,s)\, e(x,s)) + \beta \mathbb{E}\left[V^{bf}(x',s') \mid s, e\right], \text{ and}$$

$$V^{lf}(x,s) = \tau^b(x,s) + \frac{1}{1+r}\mathbb{E}\left[V^{lf}(x',s') \mid s, e\right];$$

where $\tau^b(x,s) = \theta f(n^b(x,s)) - G - c^b(x,s)$. The policy functions defining the *Fund* contract are given by the first-order conditions of (9). In particular, $c^b(x,s)$ and $n^b(x,s)$ satisfy

$$u'(c^b(x,s)) = \frac{1+v_l(x,s)}{1+v_b(x,s)}\frac{1}{x} \quad \text{and} \quad \frac{h'(1-n^b(x,s))}{u'(c^b(x,s))} = \theta f'(n^b(x,s))$$

The effort policy $e^b(x,s)$ is more complex since the first-order condition with respect to e is:

$$\begin{aligned} & x\left[(1+v_b(x,s))v'(e(x,s) + \widetilde{\xi}(x,s)v''(e(x,s)\right] \\ = \;& \frac{1+v_l(x,s)}{1+r}\sum_{s'|s}\pi^\theta(\theta'|\theta)\frac{\partial \pi^G(G'|G,e(x,s))}{\partial e}\left[x'V^{bf}(x',s') + V^{lf}(x',s')\right] \\ & + \frac{1}{1+r}\sum_{s'|s}\pi^\theta(\theta'|\theta)\pi^G(G'|G,e)\eta\widetilde{\xi}(x,s)x\left[\frac{\partial^2\pi^G(G'|G,e(x,s))/\partial e\partial e}{\pi^G(G'|G,e(x,s))} - \frac{\left(\partial\pi^G(G'|G,e(x,s))/\partial e\right)^2}{\pi^G(G'|G,e(x,s))^2}\right]V^{bf}(x',s'). \end{aligned} \tag{10}$$

Notice that if the *incentive constraint* is not binding (i.e. $\widetilde{\xi}(x,s) = 0$), then (10) reduces to:

$$x'v'(e(x,s) = \frac{1}{1+r}\sum_{s'|s}\pi^\theta(\theta'|\theta)\frac{\partial\pi^G(G'|G,e(x,s))}{\partial e}\left[x'V^{bf}(x',s') + V^{lf}(x',s')\right]. \tag{11}$$

Dividing (10) by x and rearranging terms becomes:

$$\begin{aligned} & (1+v_b(x,s))v'(e(x,s) + \widetilde{\xi}(x,s)v''(e(x,s) \\ = \;& \sum_{s'|s}\pi^\theta(\theta'|\theta)\frac{\partial\pi^G(G'|G,e(x,s))}{\partial e}\left[\beta\left(1+v_b+\varphi(G' \mid G,e)\right)V^{bf}(x',s') + \frac{1}{1+r}\frac{1+v_l(x,s)}{x}V^{lf}(x',s')\right] \\ & +\beta\sum_{s'|s}\pi^\theta(\theta'|\theta)\pi^G(G'|G,e)\widetilde{\xi}(x,s)\left[\frac{\partial^2\pi^G(G'|G,e(x,s))/\partial e\partial e}{\pi^G(G'|G,e(x,s))} - \frac{\left(\partial\pi^G(G'|G,e(x,s))/\partial e\right)^2}{\pi^G(G'|G,e(x,s))^2}\right]V^{bf}(x',s') \end{aligned}$$

The left-hand side of (10) is the social marginal cost of effort, while the right-hand side are the conditional rewards (or punishments) corresponding to the different states $s'(\cdot, G')$. The contract must stablish a system of punishments and rewards such that (10) is satisfied, while the *intertemporal incentive constraint* (6) is also satisfied. Therefore, if we substitute (6) into (10) the remaining equation must also be satisfied; that is,

$$\begin{aligned} & \widetilde{\xi}(x,s)v''(e(x,s) \\ = \;& \sum_{s'|s}\pi^\theta(\theta'|\theta)\frac{\partial\pi^G(G'|G,e(x,s))}{\partial e}\left[\beta\varphi(G' \mid G,e)V^{bf}(x',s') + \frac{1}{1+r}\frac{1+v_l(x,s)}{x}V^{lf}(x',s')\right] \\ & +\beta\sum_{s'|s}\pi^\theta(\theta'|\theta)\pi^G(G'|G,e)\widetilde{\xi}(x,s)\left[\frac{\partial^2\pi^G(G'|G,e(x,s))/\partial e\partial e}{\pi^G(G'|G,e(x,s))} - \frac{\left(\partial\pi^G(G'|G,e(x,s))/\partial e\right)^2}{\pi^G(G'|G,e(x,s))^2}\right]V^{bf}(x',s'), \end{aligned}$$

which, using the definition of $\varphi(G' \mid G, e)$, simplifies to the following equality between the 'non-accounted' marginal cost of effort and the 'non-accounted' expected marginal benefit of effort:

$$
\begin{aligned}
& \mathrm{NMC(s)} \equiv \widetilde{\xi}(\mathrm{x,s})\mathrm{v}''(\mathrm{e(x,s)} \\
& = \frac{1}{1+r}\sum_{s'|s}\pi^\theta(\theta'|\theta)\pi^G(G'|G,e)\left[\widetilde{\xi}(x,s)\eta\frac{\partial^2\pi^G(G'|G,e(x,s))/\partial e\partial e}{\pi^G(G'|G,e(x,s))}V^{bf}(x',s')\right. \\
& \left. +\frac{1+v_l(x,s)}{x}\frac{\partial\pi^G(G'|G,e(x,s))/\partial e}{\pi^G(G'|G,e(x,s))}V^{lf}(x',s')\right] \\
& \equiv \frac{1}{1+r}\mathbb{E}\left[\mathrm{NMB(s')|s}\right]. \qquad (12)
\end{aligned}
$$

In order to calibrate the model, we provide more structure by assuming that, given current government liabilities G^c, there are two possible distributions of tomorrow's liabilities, $\pi^b(\cdot|G^c)$ and $\pi^g(\cdot|G^c)$, and $\pi^g(\cdot|G^c)$ first-order stochastically dominates $\pi^b(\cdot|G^c)$ for all G; in particular, there is $\zeta(e)$ with $\zeta'(e) < 0$, such that $\pi^G(G'|G^c, e) = \zeta(e)\pi^b(G'|G^c) + (1-\zeta(e))\pi^g(G'|G^c)$. Therefore,

$$\frac{\partial\pi^G(G'|G,e)}{\partial e} = -\zeta'(e)\left[\pi^g(G'|G^c) - \pi^b(G'|G^c)\right]$$

Furthermore, we also assume that $v(e) = \omega e^2$ and $\zeta(e) = \exp(-\rho e)$. In this case[10] (12) becomes:

$$
\begin{aligned}
& \widetilde{\xi}(x,s)2\omega \\
= \quad & \frac{1}{1+r}\sum_{s'|s}\pi^\theta(\theta'|\theta)\pi^G(G'|G,e)\rho\exp(-\rho e)\frac{\pi^g(G'|G)-\pi^b(G'|G)}{\pi^G(G'|G,e(x,s))}\left[\frac{1+v_l(x,s)}{1+r}\frac{1}{x}V^{lf}(x',s') - \rho\eta\widetilde{\xi}(x,s)V^{bf}(x',s')\right]
\end{aligned}
$$

4 Decentralization of the fund contract

We now show how to decentralize the optimal contract as a competitive equilibrium with endogenous borrowing constraints, which will allow us to compare the different fund contracts with the debt contract of the economy with incomplete markets. We build on the work of Alvarez and Jermann (2000) and Krueger, Lustig and Perri (2008). To make it more comparable with the incomplete market model we consider that agents trade in *state-contingent bonds (assets or securities)*. Specifically, at the beginning of a period, in state s, the borrower holds a portfolio $a(s)$ of securities (δ, κ, s), where a fraction $1-\delta$ of the portfolio matures in the period and a fraction δ pays a coupon κ and it is traded for S portfolios $a(s')$, that can be decomposed into a common portfolio that it is carried to the next period, independently of the next period state, $a'(s)$ and S insurance portfolios $\hat{a}(s')$; i.e. $a(s') = a'(s) + \hat{a}(s')$.

Note that other forms of decentralization are possible – for example, using an active management of the debt maturity structure and partial forms of default to induce state contingent contracts, as in Dovis 2016 – however our main purpose here is to have clear comparison between the two regimes and this decentralization is possibly the simplest one since $(a(s), a'(s))$ can be identified with (b, b') in state s, and $\hat{a}(s')$ corresponds to the insurance component of a special Arrow security, which provides one unit of the long-bond a in state s'.

[10] Notice that then, $\frac{\partial\pi^G(G'|G,e(x,s))}{\partial e} = \rho\exp(-\rho e)\left(\pi^g(G'|G) - \pi^b(G'|G)\right)$ and $\frac{\partial^2\pi^G(G'|G,e(x,s))}{\partial e\partial e} = -\rho^2\exp(-\rho e)\left(\pi^g(G'|G) - \pi^b(G'|G)\right)$.

4.1 The competitive equilibrium

In the market equilibrium, the borrower has a home technology that produces $\theta(s)f(n(s))$ with his own labor. The borrower has access to long term state-contingent assets and solves the following dynamic programming problem:

$$W^b(a,\, s) = \max_{(c,\, n,\, e,\, a(s'))} \left\{ U(c,\, n,\, e) + \beta \mathbb{E}\left[W^b(a(s'), s') \mid s \right] \right\}$$

$$\text{s.t. } c + q(s)\left(a'(s) - \delta a(s)\right) + \sum_{s'|s} q\left(s'|s\right) \hat{a}(s') + \tau^e(s) \leq \theta(s) f(n) - G(s) + \left(1 - \delta + \delta\kappa\right) a(s) + \tau^r(s|\ s_-)$$

$$a(s') \geq A_b\left(s'\right), \text{ where}$$

$a\left(s'\right) = a'(s) + \hat{a}\left(s'\right)$, $\sum_{s'|s} q\left(s'|s\right) \hat{a}(s') = 0$ and $q(s) = \sum_{s'|s} q\left(s'|s\right)$; that is, $a'(s) = \frac{\sum_{s'|s} q(s'|s) a(s')}{q(s)}$.

Note that, in contrast with the incomplete markets economy, $q(s)$ is independent of the amounts of securities being traded. This follows from the fact that the endogenous borrowing constraint $a(s') \geq A_b\left(s'\right)$ prevents the borrower from defaulting. We assume, without loss of generality, $a_b\left(s_0\right) = a_l\left(s_0\right) = 0$. Furthermore, $\tau^e(s)$ and $\tau^r(s|\ s_-)$ are Pigouvian taxes and rewards, respectively; where $\tau^r(s|\ s_-)$ denotes a reward in state s conditional on the state the previous period being s_-. We assume they satisfy the following non-arbitrage condition: $\tau^e(s) = \sum_{s'|s} Q(s'|s)\tau^r(s'|s)$, where $Q(s'|s) = \frac{q(s'|s)}{1-\delta+\delta\kappa+\delta q(s')}$ and $q(s') = \sum_{s''|s'} q(s''|s')$; as well as: $\tau^r(s_0|\ s_{-1}) = 0$. These taxes are designed to align the individual and the social intertemporal values of effort, given that the individual choice is determined by:

$$v'(e) = \beta \sum_{s'|s} \frac{\partial \pi(s'|s,\, e)}{\partial e} W^b(a'(s'), s').$$

In other words, Pigouvian taxes and rewards internalize the social value of effort into the borrower's intertemporal budget constraint. However, given our assumptions, $\tau^e(s) = \sum_{s'|s} Q(s'|s)\tau^r(s'|s)$ and $\tau^r(s_0|\ s_{-1}) = 0$, they cancel-out in the borrower's present value constraint and, as later we show, in the determination of the endogenous participation constraints; therefore, they do not affect the equilibrium allocations. We maintain the Pigouvian taxes and rewards in our exposition to show, in the next subsection, that *all* the equations characterizing a constrained efficient fund allocation can be decentralied.

The choice of consumption and assets determines the following Euler condition:

$$q\left(s'|s\right) \geq \beta\pi\left(s'|s\right) \frac{u'\left(c\left(s'\right)\right)}{u'\left(c\left(s\right)\right)} \left[1 - \delta + \delta\kappa + \delta \sum_{s''|s'} q\left(s''|s'\right) \right],$$

with equality if $a_b\left(s'\right) > A_b\left(s'\right)$. Alternatively, $Q(s'|s) \geq \beta\pi\left(s'|s\right) \frac{u'(c(s'))}{u'(c(s))}$. This latter condition, together with the endogenous borrowing constraint and the boundedness of $A_b\left(s'\right)$, which is given by $W^b(A_b\left(s\right), s) = V^a(s)$, guarantee that the borrower's present value budget constraint is satisfied.

The lender receives the coupon and can trade long-term assets and receives the net Piguvian revenues:

$$W^l(a,\, s) = \max_{(c,\, a(s'))} \left\{ c + \frac{1}{1+r}\mathbb{E}\left[W^l(a(s'), s') \mid s\right] \right\}$$

$$\text{s.t. } c + q(s)\left(a'(s) - \delta a(s)\right) + \sum_{s'|s} q\left(s'|s\right) \hat{a}(s') = \left(1 - \delta + \delta\kappa\right) a(s) + \tau^e(s) - \tau^r(s|\ s_-)$$

$$a\left(s'\right) \geq A_l\left(s'\right)$$

The corresponding Euler conditions is:

$$q\left(s'|s\right) \geq \frac{1}{1+r}\pi\left(s'|s\right)\left[1 - \delta + \delta\kappa + \delta \sum_{s''|s'} q\left(s''|s'\right)\right],$$

or alternatively $Q(s'|s) \geq \frac{\pi(s'|s)}{1+r}$, with equality if $a_l\left(s'\right) > A_l\left(s'\right)$, where $W^l(A_l\left(s\right), s) = Z$, which guarantees that the lender's present value budget constraint is satisfied.

In particular, in equilibrium

$$\begin{aligned}
q\left(s'|s\right) &= \frac{\pi\left(s'|s\right)}{1+r} \max\left\{ \frac{u'\left(c\left(s'\right)\right)\eta}{u'\left(c\left(s\right)\right)} \left[(1-\delta+\delta\kappa) + \delta\sum_{s''|s'} q\left(s''|s'\right)\right], \left[(1-\delta+\delta\kappa) + \delta\sum_{s''|s'} q\left(s''|s'\right)\right] \right\} \\
&= \frac{\pi\left(s'|s\right)}{1+r}\left[(1-\delta+\delta\kappa) + \delta\sum_{s''|s'} q\left(s''|s'\right)\right] \max\left\{ \frac{u'\left(c\left(s'\right)\right)\eta}{u'\left(c\left(s\right)\right)}, 1 \right\} \\
&= \frac{\pi\left(s'|s\right)}{1+r}\left[(1-\delta+\delta\kappa) + \delta q\left(s'\right)\right] \max\left\{ \frac{u'\left(c\left(s'\right)\right)\eta}{u'\left(c\left(s\right)\right)}, 1 \right\}, \text{ i.e.} \\
Q\left(s'|s\right) &= \frac{\pi\left(s'|s\right)}{1+r} \max\left\{ \frac{u'\left(c\left(s'\right)\right)\eta}{u'\left(c\left(s\right)\right)}, 1 \right\}.
\end{aligned}$$

Therefore,

$$\frac{1}{1+r(s)} \equiv Q\left(s\right) \equiv \sum_{s'|s} Q\left(s'|s\right) \geq \frac{1}{1+r}$$

which results in a **negative spread**: $r(s) - r \leq 0$.

Let $c_b\left(a_b, s\right)$, $n(a_b, s)$, $e(a_b, s)$ and $a_b\left(a_b, s'\right)$, and $c_l\left(a_l, s\right)$ and $a_l\left(a_l, s'\right)$ be the optimal policies of the borrower and the lender, respectively. Market clearing implies that:

$$\begin{aligned}
c_b\left(a_b, s\right) + c_l\left(a_l, s\right) &= \theta(s) f\left(n(a_b, s)\right) - G\left(s\right), \\
a_b\left(a_b, s'\right) + a_l\left(a_l, s'\right) &= 0.
\end{aligned}$$

Finally, note that we have assumed that our definition of the endogeneous borrowing constraints

$$W^b(A_b\left(s\right), s) = V^a(s) \tag{13}$$

$$W^l(A_l\left(s\right), s) = Z \tag{14}$$

implies the boundedness of $A_b\left(s\right)$ and $A_l\left(s\right)$. This, in turn, implies that competitive equilibrium allocations satisfy the *high implied interest rate condition*, namely:

$$\sum_{t=0}^{\infty} \sum_{s^t} Q\left(s^t|s_0\right)\left[c\left(s^t\right) + c_l\left(s^t\right)\right] < \infty,$$

where $Q\left(s^t|s_0\right) = Q\left(s^1|s_0\right) Q\left(s^2|s^1\right) ... Q\left(s^t|s^{t-1}\right)$.

4.2 Decentralization

Now we show how a *Fund* contract can be decentralized as a competitive equilibrium with long-term assets and endogenous borrowing limits. This allow us to obtain asset prices and holdings supporting the *Fund* contract, which we can compare to the debt prices and holdings of the incomplete markets economy. It will also allow us to define the Pigou taxes and rewards that implement the efficient level of effort.

Let $(c^*(x,s), n^*(x,s), e^*(x,s), \tau^*(x,s))$ be the optimal policy allocations of the *Fund*. First, we use the allocations to price the long-term assets as follows:

$$q^*(s'|s) = \frac{1}{1+r}\pi(s'|s)\frac{u'(c(s'))\eta}{u'(c(s))}\left[1-\delta+\delta\kappa+\delta\sum_{s''|s'} q^*(s''|s')\right] \text{ if } v_b(x',s') = 0 \ \& \ v_l(x',s') \geq 0; \text{ while}$$

$$q^*(s'|s) = \frac{1}{1+r}\pi(s'|s)\left[1-\delta+\delta\kappa+\delta\sum_{s''|s'} q^*(s''|s')\right] \text{ if } v_l(x',s') = 0 \text{ and } v_b(x',s') > 0.$$

Therefore, using the Fund allocation we obtain:

$$q^*(s'|s) = \frac{1}{1+r}\pi(s'|s)\left[(1-\delta+\delta\kappa)+\delta q^*(s')\right]\max\left\{\frac{1+v_l(x',s')}{1+v_b(x',s')}\frac{1}{1+\frac{\varphi(s'|x,s)}{1+v_b(x,s)}},1\right\}, \text{ i.e.}$$

$$Q^*(s'|s) = \frac{\pi(s'|s)}{1+r}\max\left\{\frac{1+v_l(x',s')}{1+v_b(x',s')}\frac{1}{1+\frac{\varphi(s'|x,s)}{1+v_b(x,s)}},1\right\}.$$

Since we impose borrowing limits that bind exactly when the participation constraints are binding in the optimal fund contract, asset prices $q(s'|s) = q^*(s'|s)$ satisfy the Euler conditions in the competitive equilibrium characterized above. Therefore, we obtain the ***price of a long-term bond*** $q^*(s) = \sum_{s'|s} q^*(s'|s)$, the implicit interest rate $r^*(s) = \frac{1}{Q^*(s)} - 1 = \frac{1}{\sum_{s'|s} Q^*(s'|s)} - 1$, which results in a, possibly, ***negative spread***: $r^f(s^t) - r \leq 0$.

Note that both, the lender and the borrower, intertemporal participation constraints cannot be simultaneously binding, as long as there are positive expected rents to be shared. Without *moral hazard* $\varphi(s'|x,s) = 0$ and, therefore, the *negative spread* reflects the fact that the lender intertemporal participation constraint is binding; that is, $Q^*(s) > \frac{1}{1+r}$ only if $v_l(x',s') > 0$, in which case $\tau(x',s') \geq 0$, otherwise the lender could simply relax the costly constraint by lending less. However with *moral hazard*, if s' is a bad state it may be the case $\varphi(s'|x,s) < 0$, resulting in a *negative spread* even if the the lender intertemporal participation constraint is not binding

In sum, *the negative spread,* $r^f(s) - r < 0$, reflects a *the wedge* that aligns the market price with the lender unwillingness to lend in some states of the future. Furthermore, if the lender is unconstrained, the borrower must be constrained in those states which are less likely when the effort implicitely prescribed by the fund contract is exercised.

Note also that, given our assumptions, there is a one-to-one correspondence between the state variable x in the Fund problem and a in the decentralized problem, given by:

$$u'(c(a,s)) = \frac{1+v_l(x,s)}{1+v_b(x,s)}\frac{1}{x};$$

that is, if at s, a and x satisfy this one-to-one correspondence, then $c(a,s)=c^*(x,s)$, and, similarly, for the the value functions: $W^b(a,\,s)=V^{bf}(x,s)$.

We now define the Pigou taxes and rewards to show that the condition $\mathrm{NMC(s)}=\frac{1}{1+\mathrm{r}}\mathbb{E}\left[\mathrm{NMB(s'|s)}\right]$ (i.e. condition (12)) is satisfied, whenever the the non-arbitrage condition, $\tau^e(s)=\sum_{s'|s}Q^*(s'|s)\tau^r(s'|s)$ is satisfied. Let:

$$\tau^e(s)\equiv\mathrm{NMC(s)}=\widetilde{\xi}\mathrm{(x,s)v''(e(x,s)}$$

and

$$\begin{aligned}
\tau^r(s'|s) \quad &\equiv \quad \frac{\mathrm{NMB(s'|s)}}{\max\left\{\frac{u'(c(s'))\eta}{u'(c(s))},1\right\}} \\
&= \quad \left[\widetilde{\xi}(x,s)\eta\frac{\partial^2\pi^G(G'|G,e(x,s))/\partial e\partial e}{\pi^G(G'|G,e(x,s))}V^{bf}(x',s')+\frac{1+v_l(x,s)}{x}\frac{\partial\pi^G(G'|G,e(x,s))/\partial e}{\pi^G(G'|G,e(x,s))}V^{lf}(x',s')\right] \\
&\times \quad \left[\max\left\{\frac{1+v_l(x',s')}{(1+v_b(x',s'))}\frac{1}{1+\frac{\varphi(s'|x,s)}{1+v_b(x,s)}},1\right\}\right]^{-1} \\
&\leq \quad \mathrm{NMB(s'|s)}.
\end{aligned}$$

Note that:

$$\begin{aligned}
\mathrm{NMC(s)}=\tau^{\mathrm{e}}(\mathrm{s}) \quad &= \quad \sum_{\mathrm{s'|s}}\mathrm{Q^*(s'|s)\tau^r(s'|s)} \\
&= \quad \frac{1}{1+r}\sum_{s'|s}\pi(s'|s)\max\left\{\frac{u'\left(c\left(s'\right)\right)\eta}{u'\left(c\left(s\right)\right)},1\right\}\tau^r(s'|s) \\
&= \quad \frac{1}{1+r}\sum_{s'|s}\pi(s'|s)\mathrm{NMB(s'|s)}.
\end{aligned}$$

Finally, we use the intertemporal budget constraints to construct the ***asset holdings*** that make the allocations in the optimal contract satisfy the present value budget, namely:

$$\begin{aligned}
a_b\left(s^t\right) \quad &= \quad \sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t}Q^*\left(s^{t+n}|s^t\right)\left[c^*\left(s^{t+n}\right)+\tau^e(s^{t+n})-\left(\theta(s^{t+n})f\left(n^*(s^{t+n})\right)-G\left(s^{t+n}\right)+\tau^r(s^{t+n}|s^{t+n-1})\right)\right] \\
&= \quad -\sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t}Q^*\left(s^{t+n}|s^t\right)\tau^*\left(s^{t+n}\right) \\
a_l\left(s^t\right) \quad &= \quad \sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t}Q^*\left(s^{t+n}|s^t\right)c_l\left(s^{t+n}\right)=\sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t}Q^*\left(s^{t+n}|s^t\right)\tau^*\left(s^{t+n}\right) \\
a_l\left(s^t\right) \quad &= \quad -a_b\left(s^t\right).
\end{aligned}$$

In this economy, binding participation constraints provide us with the borrowing limits given by

(13) and (14). More precisely,

$$
\begin{aligned}
A_b\left(s^t\right) &= \sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t} Q^*\left(s^{t+n}|s^t\right)\left[c^*\left(s^{t+n}\right) - \left(\theta(s^{t+n})f\left(n_b^*(s^{t+n})\right) - G\left(s^{t+n}\right)\right) + \tau^r(s^{t+n}|s^{t+n-1}) - \tau^e(s^{t+n})\right] \\
&= -\sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t} Q^*\left(s^{t+n}|s^t\right)\left(\tau^*\left(s^{t+n}\right)\right) \qquad (15) \\
A_l\left(s^t\right) &= Z \\
&= \sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t}\left(\frac{1}{1+r}\right)^t \tau^*\left(s^{t+n}\right) \qquad (16)
\end{aligned}
$$

where the first equality refers to histories $\{s^{t+n}\}_{n=0}^{\infty}$ following a state s^t where the limited enforcement constraint of the borrower is binding (i.e. the borrower is indifferent between remaining in the *Fund* contract and autarky) and, similarly the last equality corresponds to histories following a state where the limited enforcement constraint of the lender, who values transfers at the risk-free interest rate, is binding.

The corresponding *recursive competitive equilibrium* for these *Fund* decentralized economies is also defined in the standard way as a set of policy functions: $c(a_b, s)$, $n(a_b, s)$, $e(a_b, s)$, $a_b'(a_b, s), \tau(a_l, s), a_l'(a_l, s)$ and value functions, W^{bf}, W^{lf}, that solve the agents problems for the corresponding Arrow security prices, $q(s'|s)$ and, finally, markets clear. In particular, as we have seen, the value functions $(W^{bf}(a_b, s), W^{lf}(a_l, s))$ are the mirror image of the value functions $(V^{bf}(x, s), V^{lf}(x, s))$, since, given that $a_l = -a_b$, the dimension of the state (co-state) is the same and, as we have seen the allocations are the same.

To conlude some *Fund* accounting is also useful. Paralleling the discussion of the *incomplete markets,* the ***primary surplus*** (primary deficit if negative) is given by

$$q(s)(a'(s) - \delta a(s)) - (1 - \delta + \delta k)\, a(s) = \tau(x, s).$$

5 Calibration

5.1 Functional Forms, Shock Processes and Parameter Values

The model period is assumed to be one year. To make the different contracts comparable, we choose the same parameter values across economies whenever this is possible.

The utility of the borrower is additively separable in consumption and leisure. In particular, we assume

$$\log(c) + \gamma\frac{(1-n)^{1-\sigma} - 1}{1-\sigma} - \omega e^2$$

The preference parameters are set to $\sigma = 0.6887$ and $\gamma = 1.4$. These are used, together with the discount factor $\beta = 0.945$, to match the average hours, in together with the volatility of consumption and hours relative to GDP. The interest rate is set to $r = 2.48\%$, the average short-term real interest rate of German. Note that this implies a different discount factor for the lender of $\frac{1}{1+r} = 0.9758$, as well as a growth rate for the relative Pareto weight of the borrower of $\eta = 0.9684$ in the optimal contract. Regarding the technology, we assume that $f(n) = n^\alpha$ with the labor share of the borrower set to $\alpha = 0.566$ to match the average labor share across the Euro Area 'stressed' countries. The

Table 1: Parameter Values

α	β	σ	γ	r	λ^i	ψ	δ	κ	Z
0.566	0.945	0.6887	1.4	0.0248	0.15	0.8099	0.814	0.083	0

Table 2: Parameters of the labor productivity process

	$\mu(\zeta)$	$\rho(\zeta)$	$\sigma(\zeta)$	P	$\zeta = 1$	$\zeta = 2$	$\zeta = 3$
$\zeta = 1$	6.35	0.93	0.02	$\zeta = 1$	0.90	0.10	0.00
$\zeta = 2$	6.94	0.92	0.01	$\zeta = 2$	0.06	0.87	0.07
$\zeta = 3$	7.09	0.81	0.02	$\zeta = 3$	0.01	0.08	0.91

$\omega(e)$ function determining how effort increases the likelihood of good realizations of the G^c shock is $-\exp(-\rho e)$. The participation constraint of the lender in the *Fund* contract is set to $Z = 0$, a very tight level. Finally, the probability that the borrower comes back to the market upon default is set it to $\lambda^i = 0.15$ in the incomplete market model with default, while we assume that *Fund-exit is irreversible*, therefore we set $\lambda^f = 0$. Furthermore, in both models, the default penalty takes the form

$$\theta^p = \begin{cases} \bar{\theta}, & \text{if } \theta \geq \bar{\theta} \\ \theta, & \text{if } \theta < \bar{\theta} \end{cases} \quad \text{with } \bar{\theta} = \psi \mathbb{E}\theta,$$

where $\psi = 0.8099$. The latter two parameters, together with the discount factor β are chosen to match jointly the PIIGS average debt to GDP ratio, spread level and spread volatility. Finally, the parameters of the long term bond (δ, κ) are set to $\delta = 0.814$ and $\kappa = 0.083$ to match the average maturity and the average coupon rate (coupon payment to debt ratio) of long term debt. The following table summarizes all the parameter values.

The log labor productivity $\log \theta$ is assumed to be a Markov regime switching (MRS) AR(1) process. In our calibration, we fit the labor productivity $\log(\theta_{it})$ of five PIIGS countries to the following panel MRS AR(1) model:

$$\log \theta_{it} = (1 - \rho(\zeta_{it}))\mu(\zeta_{it}) + \rho(\zeta_{it}) \log \theta_{it} + \sigma(\zeta_{it})\varepsilon_{it},$$

where $\zeta_{it} \in \{1, \ldots, R\}$ denotes the regime of country i at time t, $\mu(\zeta_{it})$, $\rho(\zeta_{it})$, and $\sigma(\zeta_{it})$ are functions of the regime, and $\varepsilon_{it} \overset{\text{iid}}{\sim} N(0, 1)$. The country specific regime s_{it} is independent in the cross-section, and follows a Markov chain over time, with an $R \times R$ regime transition matrix P.

Since our model does not have any capital accumulation, we first calculate time series for the labor productivity data for the 5 Euro Area 'stressed' countries. We then estimate the model above by adapting the expectations maximization (EM) algorithm outlined in Hamilton (1990) to our setup, combined with a more efficient procedure of Hamilton (1994) to calculate the smoothed probabilities of latent regimes. We set $R = 3$ for the panel MRS model in our estimation. Because the likelihood function of the model is highly nonlinear, the EM algorithm of likelihood maximization may be stuck at a local maximum. To overcome this potential deficiency, we randomize the initial point in the parameter space for 1,000 times. The estimated parameters of the Markov Switching Process are displayed below:

Finally, the process is then discretized into a 27-state Markov chain, with 9 values in each regime.

In the benchmark calibration without moral hazard, we consider a simple specification for the cyclical component G_c of the government consumption shock G. In particular, G_c has a state space of $\mathcal{G}_c = \{G_c(1), G_c(2), G_c(3)\}$, with $G_c(1) > G_c(2) > G_c(3)$, and the transition matrix for G_c is pinned down by two parameters[11]:

$$\pi^{G_c} = \begin{bmatrix} \phi & \frac{2}{3}(1-\phi) & \frac{1}{3}(1-\phi) \\ 2\eta & \phi & 1-\phi-2\eta \\ \eta & 1-\phi-\eta & \phi \end{bmatrix}$$

The parameters of the transition matrix are set to $\phi = 0.965$ and $\eta = 0.015$. These parameters, together with the state space for the shock, are used to match several moments of current government expenditures, such as the G to GDP ratio, the persistence of the observed government consumption, and the relative volatility of government consumption with respect to output. The resulting transition matrix and government shock values of G^c are given below:

$$\pi^{G_c} = \begin{bmatrix} 0.9650 & 0.0233 & 0.0117 \\ 0.0300 & 0.9650 & 0.0050 \\ 0.0150 & 0.0200 & 0.9650 \end{bmatrix}$$

$$G^c = \begin{bmatrix} 0.038 & 0.029 & 0.025 \end{bmatrix}$$

For the iid component G_d to government expenditures, we simply assume that is uniformly distributed over $[-\bar{m}, \bar{m}] = [-0.0005, 0.0005]$. In particular, we discretize G_d into $N_d = 11$ equally spaced grid points $\{G_d(1), \ldots, G_d(N_d)\}$ over the previous interval, and set $\Pr(G_d(i)) = 1/N_d$ for all i. Using G_c and the discretized values of G_d, the discretized G shock can be constructed according to:

$$G_{(i-1)N_c+j} = G_d(i) + G_c(j), \quad i = 1, \ldots, N_d, j = 1, \ldots, N_c,$$

where $N_c = 3$. Moreover, with some slight abuse of notation, we define the transition matrix π^G of the discretized G shock to be the Kronecker product of two matrices:

$$\pi^G = \pi^{G_c} \otimes \pi^{G_d},$$

where π^{G_d} is simply an $N_d \times N_d$ matrix with all entries equal to $1/N_d$. Defining π^G this way follows directly from the fact that G_c is independent of G_d. As noted by Chatterjee and Eyigungor (2012), the iid component improve considerably the convergence properties of the model with incomplete markets and default. We also use it to match, together with G^c, the relative volatility of government expenditures.

When moral hazard is effective, we assume that effort only affects the cyclical government liabilities G^c. Given the current G^c there are two possible distributions of tomorrow's $\pi^b(\cdot|G^c)$ and $\pi^g(\cdot|G^c)$, and $\pi^g(\cdot|G^c)$ first order stochastically dominates $\pi^b(\cdot|G^c)$ for all G, with $\pi^G(G'|s,e) = \pi^G(G'|G^c,e)$ and

$$\pi^G(G'|G^c,e) = \exp(-\rho e))\pi^b(G'|G^c) + (1-\exp(-\rho e))\pi^g(G'|G^c)$$

Recall that that government expenditure is ordered in a decreasing way. This implies that increasing effort, ceteris paribus, increases the probability of low government expenditure. Note that this functional form implies simple expressions for $\frac{\partial \pi^G(G'|G,e)}{\partial e}$ and $\frac{\partial^2 \pi^G(G'|G,e)}{\partial e^2}$ as follows:

$$\frac{\partial \pi^G(G'|G,e)}{\partial e} = \rho \exp(-\rho e)[\pi^g(G'|G) - \pi^b(G'|G)]$$

[11]See Appendix for further details.

and

$$\frac{\partial^2 \pi^G(G'|G,e)}{\partial e^2} = -\rho^2 \exp(-\rho e)[\pi^g(G'|G) - \pi^b(G'|G)]$$

5.2 Data Sources and Measurement

The primary data source we use is the AMECO dataset. We use annual data for the 5 Euro Area 'stressed' countries, and except for a few series, the sample coverage is 1980–2015. Table 5 provides a summary of the data sources and definitions. We construct model consistent measures based on the raw data. In what follows, we detail on the sources and measurement methods.

5.2.1 National accounts variables

For the aggregate output Y_{it} and government consumption expenditure G_{it} of each country, we use directly the corresponding data series from AMECO over 1980–2015, measured in constant prices of 2010 euros. Since there is no capital accumulation in the model, we interpret consumption in the model as standing for private absorption, and define the model consistent measure in the data as the sum of the private consumption and gross capital formation. For the aggregate labor input n_{it}, we use two series from AMECO, the aggregate working hours H_{it} and the total employment E_{it} of each country over the period 1980–2015. We calculate the normalized labor input according to $n_{it} = H_{it}/(E_{it} \times 5200)$, assuming 100 hours of allocatable time per worker per week. However, for most parts of the data moments computations, we use H_{it} directly, since the per worker annual working hours do not show a significant cyclical pattern and both the level and the trend do not affect the computation of the moments.

5.2.2 Government bond variables

We use the end-of-year government debt to GDP ratios in AMECO to measure the indebtedness of the Euro Area 'stressed' countries. The government debt is defined as the general government consolidated gross debt. This is conceptually different from the debt in the model, which corresponds to national debt more closely. nevertheless, we use the gross debt measure, as it provides a consistent measure across countries and is arguably an upper limit on the indebtedness of the government.

We use the nominal long-term bond yields in AMECO to measure the nominal borrowing costs of the Euro Area 'stressed' countries, and use short-term interest rates in German to measure the funding cost of international investors. The risk-free rate is measured as the real short-term interest rate of Germany, which equals to the average of the nominal rate minus GDP deflator from 1980–2015. To arrive at a meaningful measure of the *real* spread, i.e., a spread unaffected by expected inflation hence rightly reflect 'stressed' countries' credit risk, we split the sample into to parts by the introduction of the euro. For the first part, 1980–1998, we use spot and forward exchange rates to convert German nominal risk free rate into each stressed country's local currency, hence deriving a synthetic local currency risk free rate, and then take the difference between the local nominal long-term bond yield with the synthetic risk free rate. Since the synthetic risk free rate is denominated in the local currency as well, so it is subject to the same inflation expectations as the long-term bond yield, and consequently, the difference is equivalent to the real spread. For the second part, 1998–2015, we can directly use the spread between the 'stressed' countries' long-term bond yields and

German short-term interest rates, since all rates are denominated in the euro, hence subject to the same inflation expectation.

The information on the maturity structure of the government debt for the Euro Area 'stressed' countries is not comprehensive. We are able to find average years to maturity for the five countries from 3 sources.[12] The overall time coverage is unequal across countries: 1998–2010 and 2014–2015 for Ireland, 1998–2015 for Greece, 1991–2015 for Spain, 1990–2015 for Italy, and 1995–2015 for Portugal.

5.2.3 Fiscal positions

For the model, the theoretically consistent measure of the primary surplus is simply $y - c - G$, i.e., the total saving of the economy. Recall that the primary surplus is defined as government surplus minus interest payments. Alternatively, by the government's budget constraint, the primary surplus can be expressed as the net lending by the government, i.e., the difference between revenue of newly issued debt and payments on interests and retiring debt. For the economy with incomplete markets and default we are considering, this equals to $q_t(b_{t-1} - b_t) - (1 - \delta + \delta\kappa)b_t$, and by the economy's budget constraint, the last expression is just equal to $y_t - c_t - G_t$, which is the measure we use for primary surplus in the model.

To be consistent with the model, we also measure the primary surplus in the data according to the last expression. Since c_t is already measured as the private absorption, i.e., sum of the private consumption and gross capital formation, the empirical measure of the primary surplus is equivalent to the net export by the national accounting identity.

5.2.4 Labor share

We use various data series from AMECO to construct the labor share of annual output for each of the Euro Area 'stressed' countries over the period 1980–2015. First, we use nominal compensation to employees of the total economy in AMECO (labeled by UWCD) to measure the labor income for employees. Second, to measure the labor income for self-employed people, we take the difference between two AMECO series, UOGD and UQGD, where the former is gross operating surplus and the latter is the same measure net off imputed compensation for the self-employed population. We define the total labor income as the sum of the labor income for employees and self-employed, i.e., UWCD + UOGD − UQGD. Finally, the labor share is calculated as the ratio of labor income to nominal GDP.

5.2.5 Labor productivity

Given the production function, $y = \theta n^{\alpha}$, we measure the labor productivity of country i at time t according to $\theta_{it} = Y_{it}/H_{it}^{\alpha}$, or equivalently, $\log \theta_{it} = \log y_{it} - \alpha \log n_{it}$. Note that we use a common α for all ' Euro Area 'stressed' countries. Let $\hat{\theta}_{it}^{o}$, or $\log \hat{\theta}_{it}^{o}$, denote the original measured level for labor productivity. To compute the data moments involving the labor productivity, we use the HP-filter to detrend the sample productivity $\{\log \hat{\theta}_{it}^{o}\}$. Moreover, as described earlier, we use a Markov regime switching model to estimate the productivity process. Before taking the data to the model, we adjust the original sample in the following two steps:

[12] See the appendix for more details.

1. We take out a common linear time trend in the $\{\log \hat{\theta}_{it}^{o}\}$ series.
2. After detrending, we further standardize $\{\log \hat{\theta}_{it}^{o}\}$ for each i so that the resulting series has the same sample mean and volatility over i. This is to prevent the level and volatility differences in $\{\log \hat{\theta}_{it}^{o}\}$ across i to induce spurious regime switching behavior in the estimation process.

We denote the adjusted sample productivity by $\{\log \hat{\theta}_{it}\}$, which is then used in the estimation of the MRS model discussed earlier.

6 The IMD vs. the Fund regimes without moral hazard

This section discusses **the numerical results without moral hazard** (i.e. $v(e) = 0$). We compare the incomplete markets economy with default (**IMD**) and the economy with a *Fund* with two sided limited of commitment (**2S**). We first present calibration results in Table 3 and the *policy functions* for both economies in Figures (1) - (2). To better understand how these economies work, we show representative paths of both economies, subject to the same sequence of shocks in steady state, in Figures (5) - (6).Finally, we study how both economies respond to a combined negative shock when they are in steady state: Figures (7) - (8). TFP shocks are labeled $e_i, i = 1, \ldots, 27$ where $e_i < e_{i+1}$ and G shocks are labeled $g_j, j = 1, \ldots, 3$ where $g_j > g_{j+1}$ –that is (e_1, g_1) **is the worst combination of shocks and, increasingly,** (e_{27}, g_3) **is the best combination of shocks.**

6.1 Calibration results

The following Table 3 provides an exhaustive account of our benchmark calibration for the Euro Area 'stressed' countries with the incomplete markets economy defaultable debt (IMD) and also the comparison of these economies with the Fund economy, subject to the same shock processes that the ones calibrated for the IMD economy. Note that our IMD economy matches remarkably well most most moments, with the notable exception of the behaviour of the *primary surplus* – both, its mean and its correlation with output. However, this seems to be more a problem of the Euro Area 'stressed' countries than of our model, since our IMD economy, even with its default events (consistent with the level of debt to GDP and the bond spread), seems to be more efficient – its fiscal policy more countercyclical – than the observed economies.

The quantitative comparison between the IMD economy and the economy with a Fund is striking. Even if the fund contract is designed to prevent persistent redistribution from the fund to the borrower country, the amount of average debt is almost two and one-half times higher with the fund and the contrast even larger when primary surpluses are compared and, consistently, the fund implements a strong counter-cyclical fiscal policy in comparison with the IMD, no to mention in relation with the pro-cyclical budget policies of the 'stressed' countries.

Table 3: Benchmark calibration with IMD and comparison with Fund

Moments	Data	IMD	Fund
Mean			
Debt to GDP ratio	77.29%	76.56%	186.65%
Real bond spread	3.88%	3.76%	-0.02%
G to GDP ratio	20.18%	19.62%	19.31%
Percentile: 1 & 99	[13.48%, 32.79%]	[11.56%, 33.02%]	[10.65%, 36.77%]
Primary surplus to GDP ratio	-0.78%	1.30%	3.57%
Fraction of working hours	36.74%	37.28%	38.09%
Volatility			
$\sigma(C)/\sigma(Y)$	1.49	1.47	0.35
$\sigma(N)/\sigma(Y)$	0.92	0.69	0.62
$\sigma(G)/\sigma(Y)$	0.91	0.86	0.53
$\sigma(PS/Y)/\sigma(Y)$	0.65	0.80	0.94
σ(real spread)	1.53%	0.93%	0.21%
Correlation			
$\rho(C, Y)$	0.88	0.76	0.57
$\rho(N, Y)$	0.67	-0.13	0.94
$\rho(PS/Y, Y)$	-0.29	0.11	0.95
$\rho(G, Y)$	0.35	0.07	0.03
$\rho(\text{real spread}, Y)$	-0.35	-0.29	-0.05
$\rho(G, \theta)$	0.13	0.00	0.00
$\rho(G_t, G_{t-1})$	0.94	0.94	0.94

6.2 Policy Functions

The core of the analysis is given by the study of the different optimal policy functions. Figures (1) and (2) display the policy functions for the main variables for the incomplete markets economy with default **IMD** and for the **Fund**, as function of the level of debt for selected (intermediate) values of shocks ($s = (\theta, G) = (e, g)$).

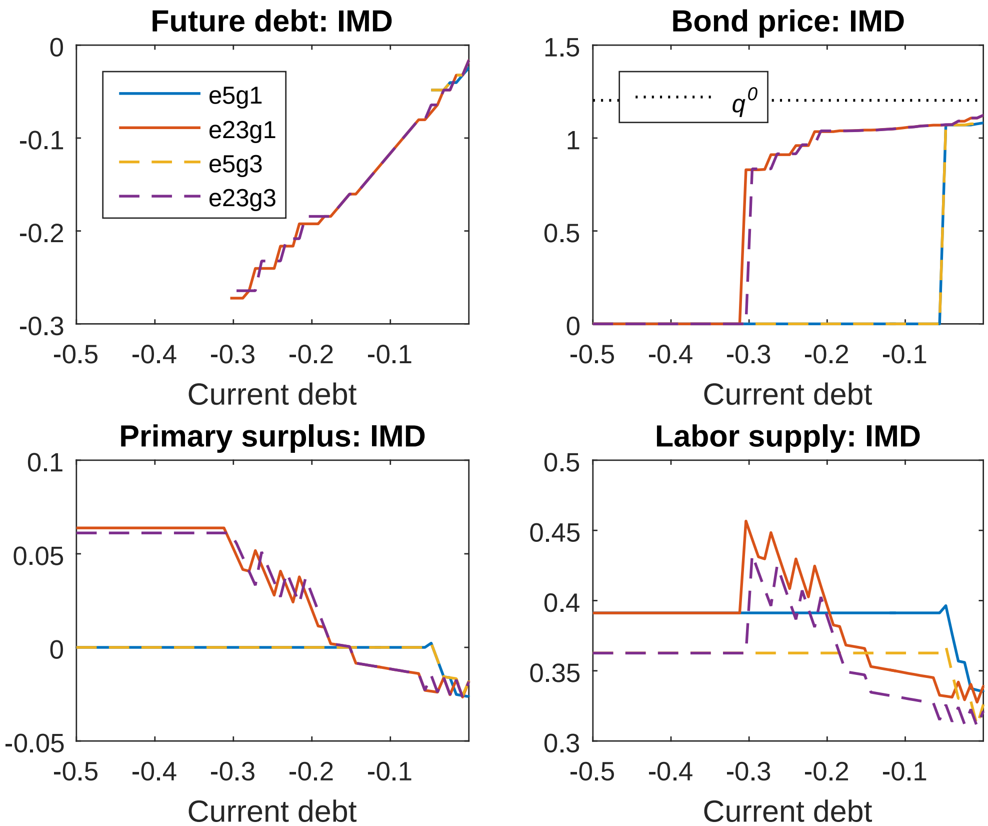

Figure 1: IMD policy functions.

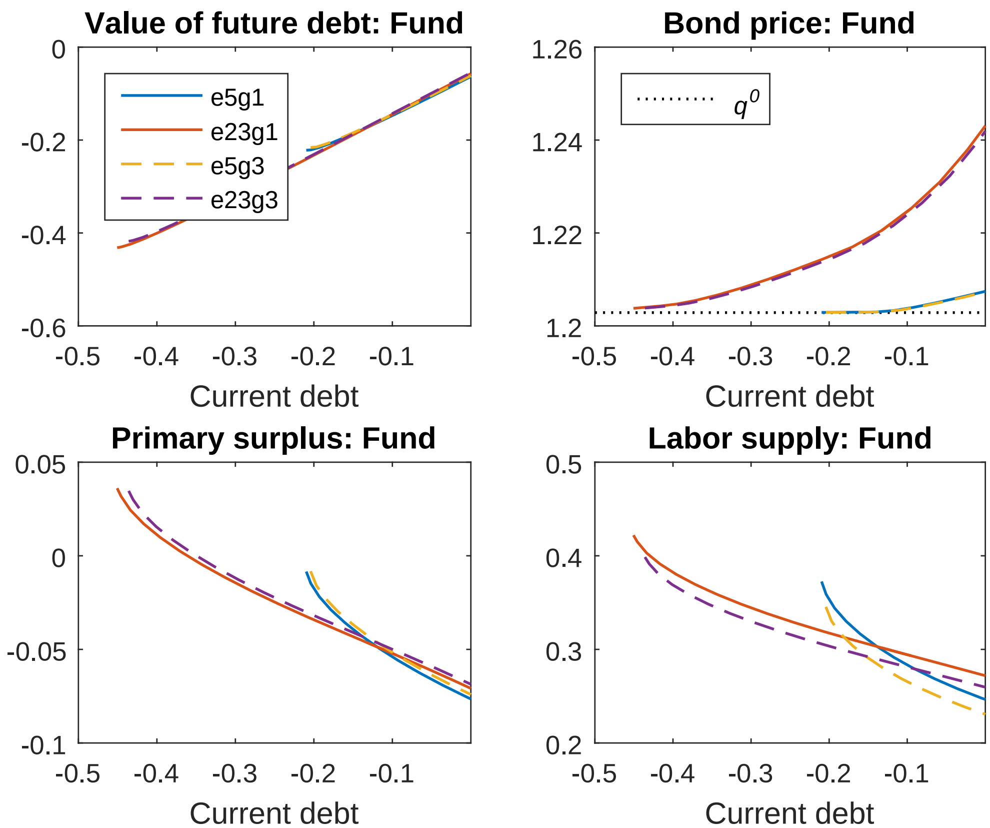

Figure 2: Fund policies as functions b.

The left panels of Figures (1) and (2) show the end-of-period debt and primary surplus policies for the **IMD** economy and the **Fund** as functions of (b, s) and (a, s), respectively. As it can be seen in the upper-left panel, for a relatively bad state $(e5g_j)$, the IMD economy only allows a minimum level of debt, while more is borrowed in the Fund, this is true at any state – as it is also true that, within a regime, more can be borrowed in better states. Furthermore, as it can also be seen in the lower-left panel, in the relatively good state the IMF economy requires to run a (positive) primary surplus for levels of debt for which the Fund has a primary deficit; both left panels show the higher debt capacity of the **Fund**. The lower-right panel shows how in the IMD economy the labor supply is distorted even at values of debt below the default threshold. It also shows that even in the Fund regime the efficient allocation of labor may be distorted (e.g. at $b = -0.2$ the supply of labor is higher when productivity is lower), although the distortion is more severe in the IMD economy. It also should be noted that expenditure shocks g do not play a major role, in comparison with the productivity shocks e. Finally, the upper-right panel shows bond prices for both regimes, in relation to the riskless bond price q^o: *positive spreads* and price collapses with default as the level of debt increases (i.e. moving to the left) in the IMD economy and, in contrast, *negative spreads* as the level of debt decreases (i.e. moving to the right) in the Fund regime.

To better understand these price effects it should be noted that, in Figure (2), the **Fund** policies as functions of $(a,\, s)$ correspond – through 'decentralization'– to **Fund** policies as functions of $(x,\, s)$, which are illustrated in Figures (3) and (4) for the same (intermediate) values of shocks. The upper-left panel of Figure (3) shows the core of the Fund mechanism. In an economy without limited enforcement or moral hazard constraints a line from the origin with slope η will determine the evolution $x \to x'$, therefore the borrower's relative Pareto weight will monotonically decrease. Such a decay – $x'(x,s) = \eta x$ – is stopped by the borrower's intertemporal participation constraints, which define the horizontal lines to the left of the 'decay line' – we denote them by $\underline{x}(s)$, i.e. $x'(\underline{x}(s),s) = \underline{x}(s)$. On the other hand, the lender's limited enforcement constraints deter x' from being too high, which define the horizontal lines to the right of the 'decay line' – we denote them by $\overline{x}(s)$ – i.e. $x'(\overline{x}(s),s) = \overline{x}(s)$. In particular, if $\underline{x}(e_{27}, g_3) > \overline{x}(e_1, g_1)$ then the support of the steady-state distribution of x is $[\underline{x}(e_1, g_1),\, \underline{x}(e_{27}, g_3)]$ and the lender's participation constraint is occasionally binding.

The other panels show the asset holding and primary surplus policies, as functions of μ_b, as well as the bond price. The patterns of these policies can be traced back to the upper-left panel. Take, for example, the state (e_{23}, g_3), increasing x towards the value of the lender's participation constraint binds (above 0.2) the bond price jumps – i.e. *negative spread* – the primary surplus becomes a primary deficit – i.e. the borrower transfers to the lender – and, consequently, debt is drastically reduced, but given that in our simulations the the lender's participation value is very tight, $Z = 0$, debt is always non-negative; i.e. the lender is never in a negative asset position. Note that now it is easier to understand the price patterns shown in Figure (1): when policies are described as functions the level of debt, and states, in the **IMD** when debts are sufficiently high there are *positive spreads*, as expected; however, as we have seen in Section 4, in the **Fund** is when the lender's participation constraint is binding that there are *negative spreads* and the lender's participation constraint is binding when the relative weight of the borrower is high (Figure 3 upper-left panel), which corresponds to very low levels of borrower's debt (Figure 2 upper-right panel); note that the lower-left panel of Figure 3 links the two representations of the borrower's states x vs. a.

Going back to the upper-left panel of Figure 3, when x is decreased to the value that the borrower's participation constraint binds (around 0.15) gives, in the lower-left panel the maximum level of debt that the Fund can have at this state (circa -0.42). Figure (4) shows the Fund consumption and labor policies, as well as the value functions, for the same intermediate states. Again, the patterns of these policies and values can be traced back to the upper-left panel of Figure (3) – in particular, the consumption policy and, more smoothly, the value of the borrower mimic the Pareto weight policy and, obviously the value of the lender mirrors the value of the borrower since both share the surplus of the Fund.

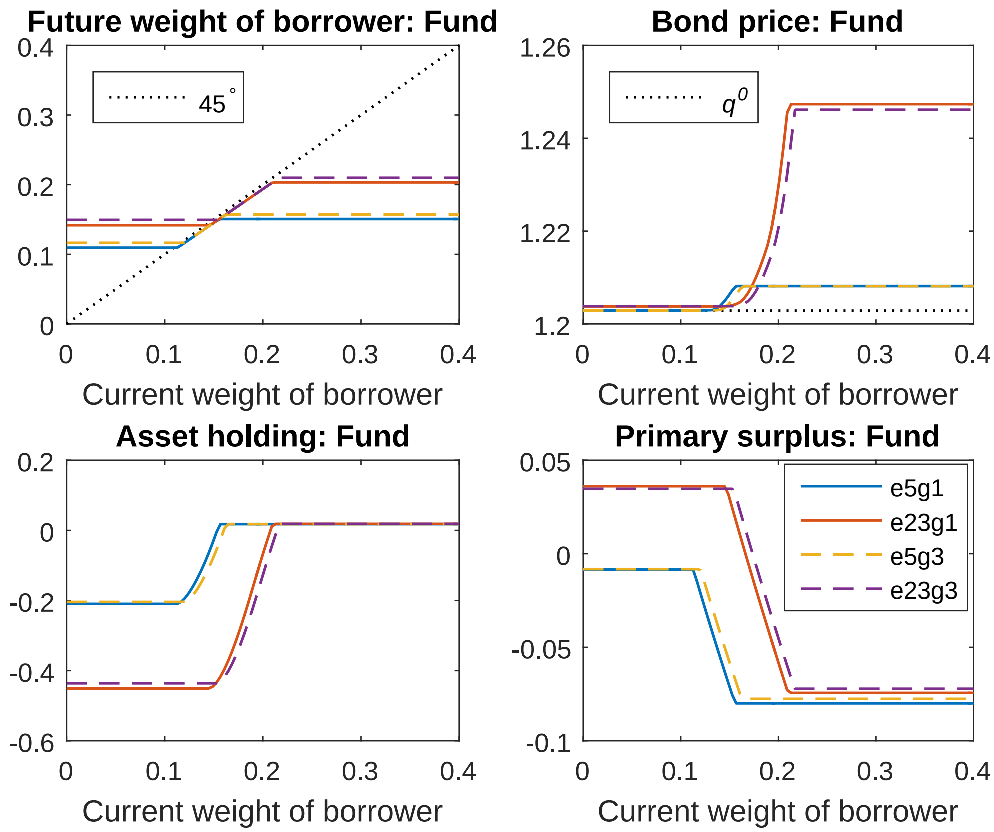

Figure 3: *Fund* policy functions: Pareto weights and assets.

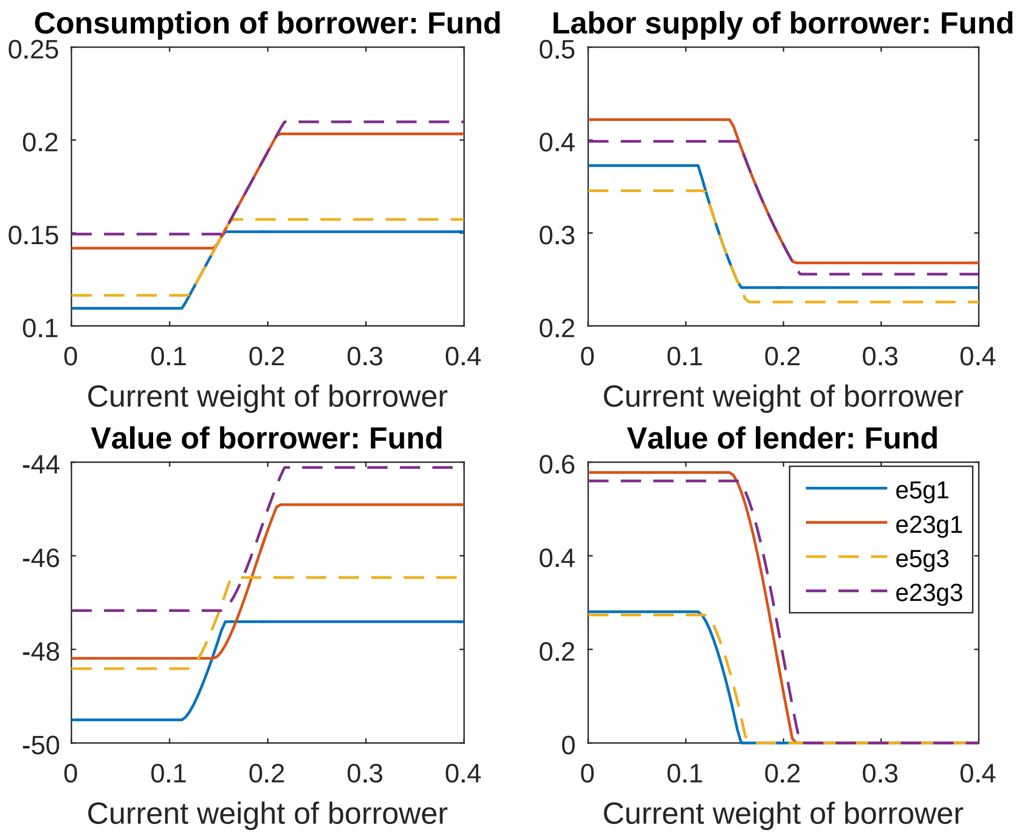

Figure 4: *Fund* policy functions: allocations and values.

6.3 Running the Fund in normal times and in times of crisis.

It is illustrative to simulate the risk-sharing outcomes of the Fund in normal times and in times of crisis. We do it here with two experiments. The first, denoted *Business Cycle Paths* – Figures (5) and (6)– is a long-run simulation at the steady state. In the second, denoted *Impulse Responses* – Figures (7) and (8) – we assume that, independently and simultaneously many independent economies are hit by negative (θ, G) shocks (e_1, g_1) but then all shocks after the initial period follow a realization of the (θ, G) stochastic process; therefore we report the average impulse response from 500 simulations. The initial endogenous conditions are randomly chosen from the stationary distribution.

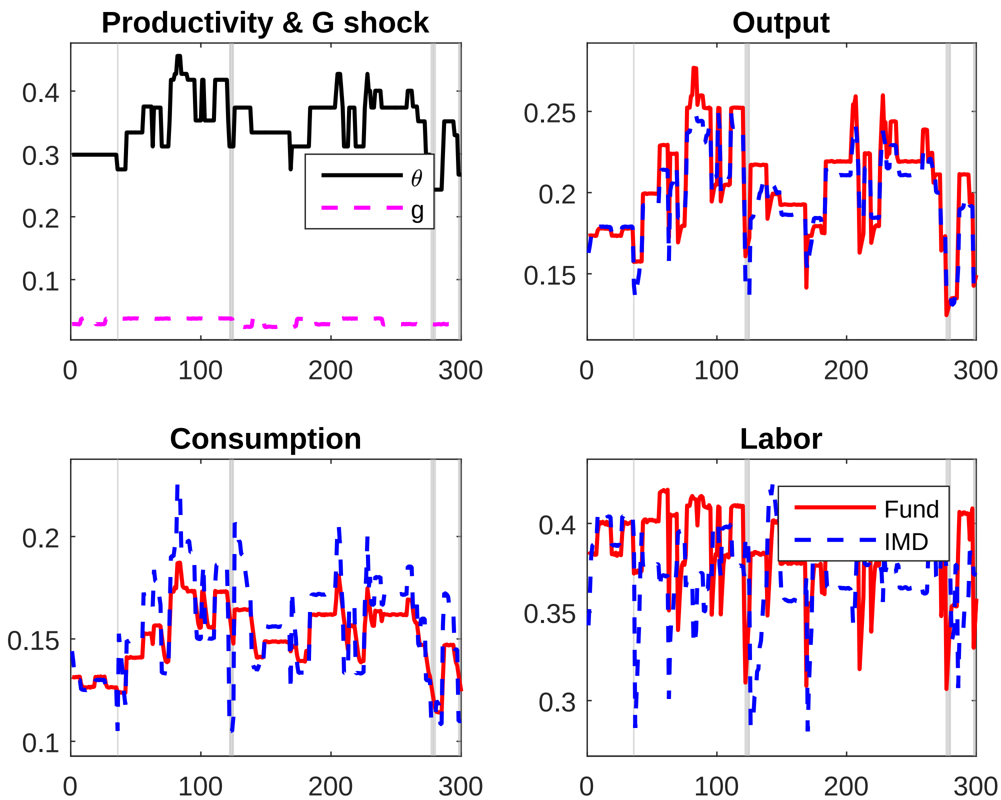

Figure 5: IMD vs. Fund Business Cycle Paths: shocks and allocations.

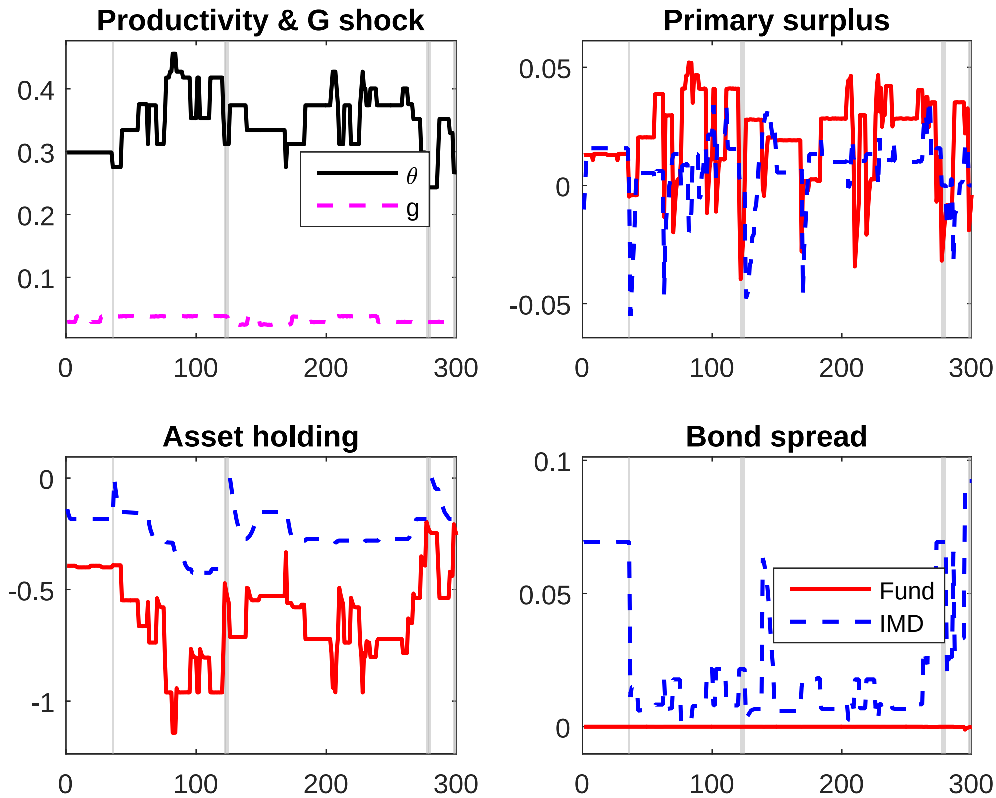

Figure 6: IMD vs. Fund Business Cycle Paths: shocks and assets.

In Figure (7), the upper-left panel shows the history of shocks for three hundred years. The grey periods correspond to periods of default in the IMD economy – defaults are associated with drops in productivity, but not all drops of productivity trigger defaults. The allocations in the IMD and Fund regimes are shown in the other panels. As it can be seen, there is more consumption smoothing in the Fund and default periods are periods of austerity where output, employment and consumption plunge. Figure (8) shows the asset allocations and prices. These panels are very revealing of why some particular productivity drops trigger defaults in the IMD economy. Just observing the evolution of shocks the first default seems puzzling since not much had happened – a small increase in government expenditures followed of a small drop in productivity, – however these were not normal times: the productivity level was medium-low but spreads were high even if there was a primary surplus: the economy was 'stressed' and transfers were countercyclical; fortunately, it was a short default episode. The other two default episodes follow a very familiar pattern: a productivity drop following relatively good years in which debt built up. Life is very different in the Fund regime: debt capacity is substantially larger and good years are years of primary surpluses; transfers from the debtor to the lender are procyclical and only a small episode of *negative spreads* happens towards the end of the series, mirroring the largest positive spread in the IMD economy.

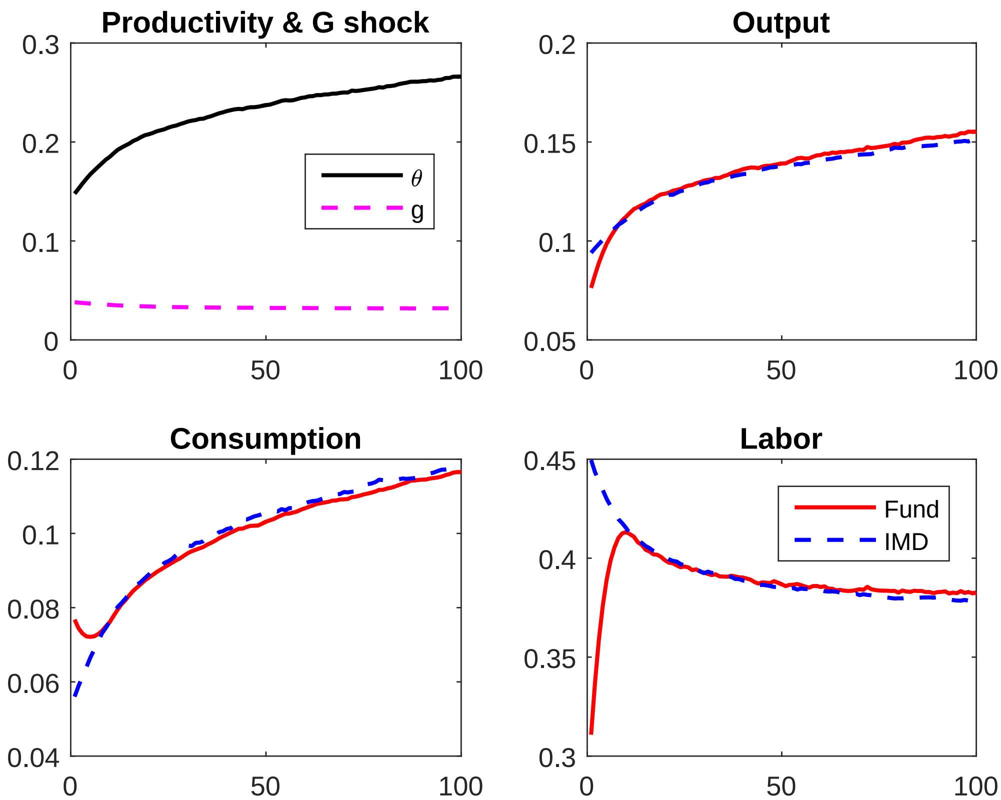

Figure 7: IMD vs. Fund: combined shock impulse-responses: allocations.

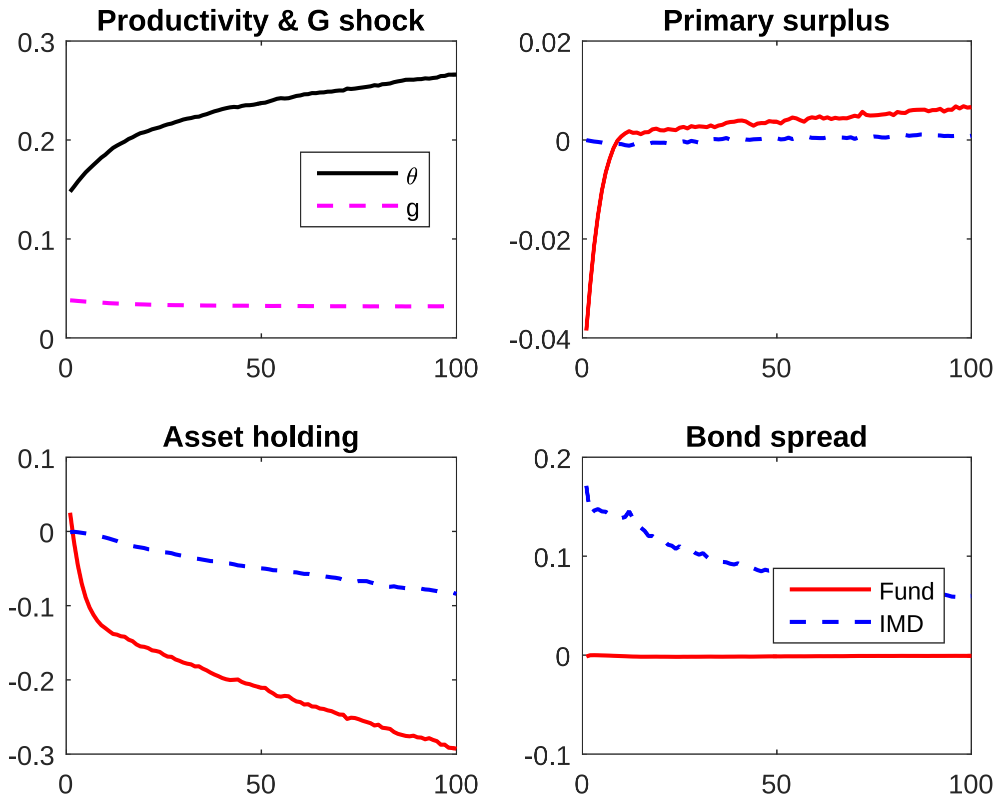

Figure 8: IMD vs. Fund: combined shock impulse-responses: assets.

As it has already been seen in default episodes, life is particularly different in times of crisis. To analyse this in more detail we induce an unexpected negative shock to our economies. Figures (7) and (8) show how the two economies react to a transitory combination of bad (θ, G) shocks. If Figure (7) everything looks very smooth is because the figure depicts average paths – for example, behind the smooth growth of output in the IMD economy there are many episodes of default, as it is reflected in the *positive spreads* of Figure (8). Nevertheless, the averages do not hide that the crisis is more severe in the incomplete markets economy with default (**IMD**) than in the economy with a *Fund*. While the crisis is a severe austerity crisis in the IMD economy, consumption is higher and labor supply lower in the Fund regime, as the immediate response to the negative shock. More remarkable is the fact that in the Fund economy a large primary deficit is allowed following the shock and, consequently, there is debt accumulation. Two facts are behind these patterns. One is the the fact that the borrower is more impatient than the lenders – being the market lenders or the Fund, – the other that the severe negative shock is a rare event. The first explains the wish to front-load consumption, which is partially satisfied with a Fund, but not in the IMD economy; the second the fact that being a rare event the borrower has ample borrowing capacity in the Fund, and none in the IMD economy, as a result in the long-run the primary surplus is higher with a Fund than in the IMD economy, showing that the former is more efficient. We now take a closer look at the relative efficiency, and the associate debt

H

Table 4: Welfare comparison at zero debt

Shocks (θ, G_c)	Welfare Gain	$(-b'/y)_{\max}$: M	$(-a'/y)_{\max}$: F
$(\theta_l, G_h) = (0.148, 0.038)$	8.90	1.71	97.42
$(\theta_m, G_h) = (0.299, 0.038)$	7.03	107.55	187.16
$(\theta_h, G_h) = (0.456, 0.038)$	4.68	217.43	336.77
$(\theta_l, G_l) = (0.148, 0.025)$	7.87	1.84	101.89
$(\theta_m, G_l) = (0.299, 0.025)$	6.56	111.40	187.93
$(\theta_h, G_l) = (0.456, 0.025)$	4.46	217.80	334.47
Average	**6.53**		

capacity, of these economies.

6.4 Welfare implications and confronting 'debt overhang' problems

Table 4 shows the increased capacity to absorb debts and the welfare gains of the Fund regime in comparison with the IMD economy. The table is constructed as follows:

- Welfare gains are in (annual) consumption equivalent terms at $b = 0$ (%).
- $-b'/y$ and $-a'/y$ are the end-of-period debt (i.e. s contingent) to output ratio in percentage for the IMD and Fund economies, respectively. Recall that for the IMD economy $-b'/y$ is also next period debt, while for the Fund economy next period debt – say, in state s' – is given by: $a(s') = a'(s) + \hat{a}(s')$; i.e. $a'(s) = \frac{\sum_{s'|s} q(s'|s)a(s')}{\sum_{s'|s} q(s'|s)}$. $a'(s)$ is the closest measure of country's indebtedness in the fund economy to the counterpart in the incomplete market with default economy.

Comparison of b'/y in the IMD economy and a'/y in the economy with a Fund shows the difference of the debt and risk-sharing capacities between the two economic regimes; in particular $(b'/y)_{\max}$ shows the maximum amount of debt to output ratio the country can have in any of the two. Only the Fund is able to absorb close to ratios close to 100% in all the states, while the capacity to absorb debts in the IMD economy is substantially smaller particularly in bad states. Welfare gains are very substantial, a consumption equivalent steady-state average welfare gain of 6.5 is a very high number and, even more relevant, a 8.9 in the worst state. Nevertheless, this is the order of magnitude that we have obtained in many other simulations that we had run as robustness check.

Table 4 suggests the following experiment: how would a highly indebted economy behave in the two different regimes? In other words, which economy will be able to deal better if there is a 'debt overhang' problem? While the table already gives an answer to this question, the details are illustrative and can be seen, for example, in Figures (9) and (10), where we started an economy with steady-state average shocks and a 100% b'/y. The IMD economy almost immediately goes into default, getting rid of its debt liabilities, and after that there is a negative widening gap on output and consumption in comparison with the defaultess economy with a Fund. In fact, the mirror of the pre-default in the IMD economy is a small *negative spread* and a subsequent increase in debt in the Fund regime. Telling figures!

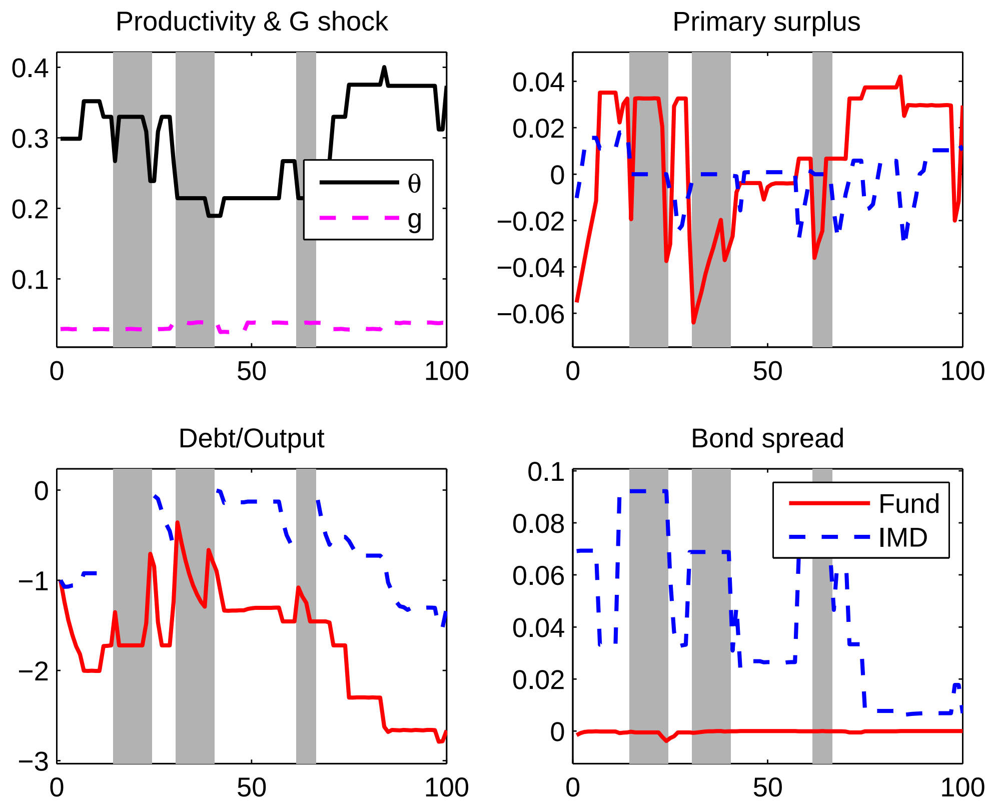

Figure 9: IMD vs. Fund in highly indebted economy: debts and spreads

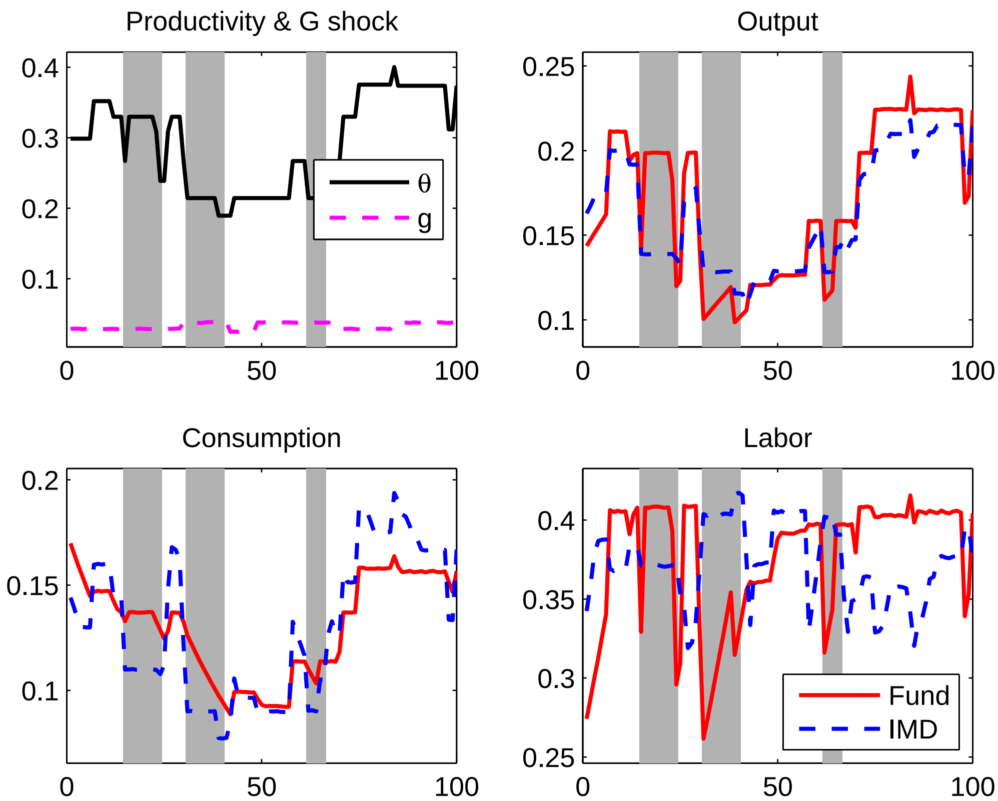

Figure 10: IMD vs. Fund in highly indebted economy: allocations

7 The IMD vs. the Fund regimes with moral hazard

In the previous simulations the distribution of the G shock was exogeneous. In this Section the distribution of G depends on the effort, e, that the borrower exercises. In particular we analyse how Fund policies change when the *moral hazard* problem is also accounted for. As we know from our calibrated exercises, that only accounted for *limited enforcement constraints*, government expenditure shocks, G, do not play a major role. Since in our formulation the unobservable effort only affects the distribution of G^c shocks, we should not expect substantial differences if the *incentive compatibility constraint* is introduced, but it is illustrative to quantify the effect. Furthermore, we want to understand how the different constraints interact and, as we know from our theory, with *moral hazard negative spreads* may be more frequent, but how much bond prices are distorted by the introduction of the *incentive compatibility* constraint? To answer these questions we study the policy functions in an economy with two-sided limited commitment (limited enforcement) with observable effort and without observable effort. Without, the economy with the Fund is the same than the economy studied in the previous Section, except in two features: first, the distribution of G is endogenous – although, the calibration is parameterised as to have similar G distributions at the steady-state, – and, second, the outside value for the borrower upon quitting the fund is autarky without the possibility to borrow in the market in the future; however, since the default penalty is very similar the difference should not be significant[13]

Figure 11 plots the Pareto weight policies with observable and unobservable effort (i.e. with *moral hazard*). With observable effort, as in the previous analysed case of an exogenous G_t process, there is no *incentive compatibility* constraint, while with unobservable effort the contract must account for the $\varphi(G_{t+1}|G_t)$ multiplier (the effect of this is shown for $g1$ and $g3$). As it can be seen, there is a small spread for the participation constraints.

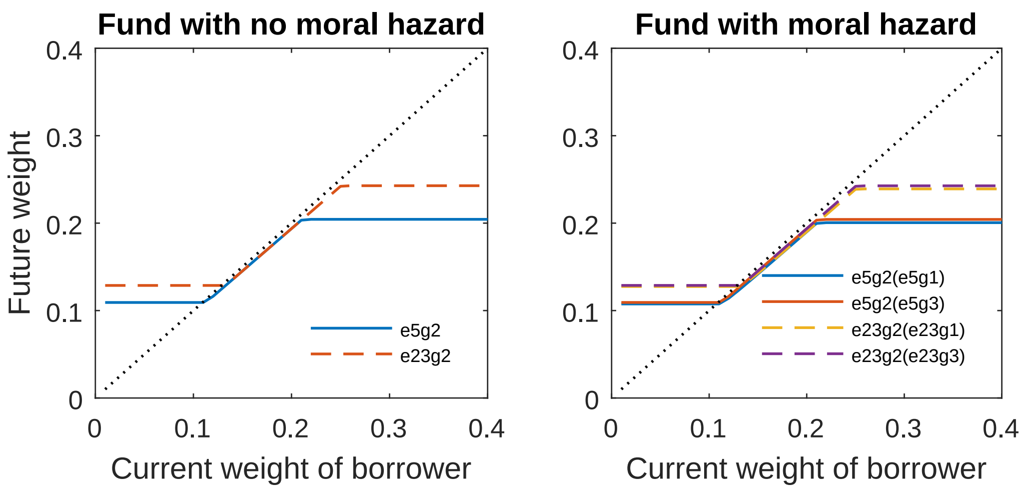

Figure 11: Pareto weight policies with exogenous and non-observable endogenous effort.

Figure 12 shows the effort policies. The differences 'at the participation constraints' are noticeable – in particular with moral hazard, effort, conditional on a relatively good ($e23g2$) and bad ($e5g2$) state, varies more when the lender's participation constraint is binding and less when the borrower's participation constraint is binding. The effect of binding *incentive compatibility* constraints – in particular, 'at, and near, the participation constraints'- translates into *negative spreads* with moral hazard that do not exist when effort is observable, as it is shown in the bottom two panels of Figure 12. In sum, the main differences of introducing *moral hazard* are the distortions created by the interplay between participation and incentive compatibility constraints and in the existence of more frequent *negative spread* episodes.

[13]This difference on default values will be eliminated in future versions.

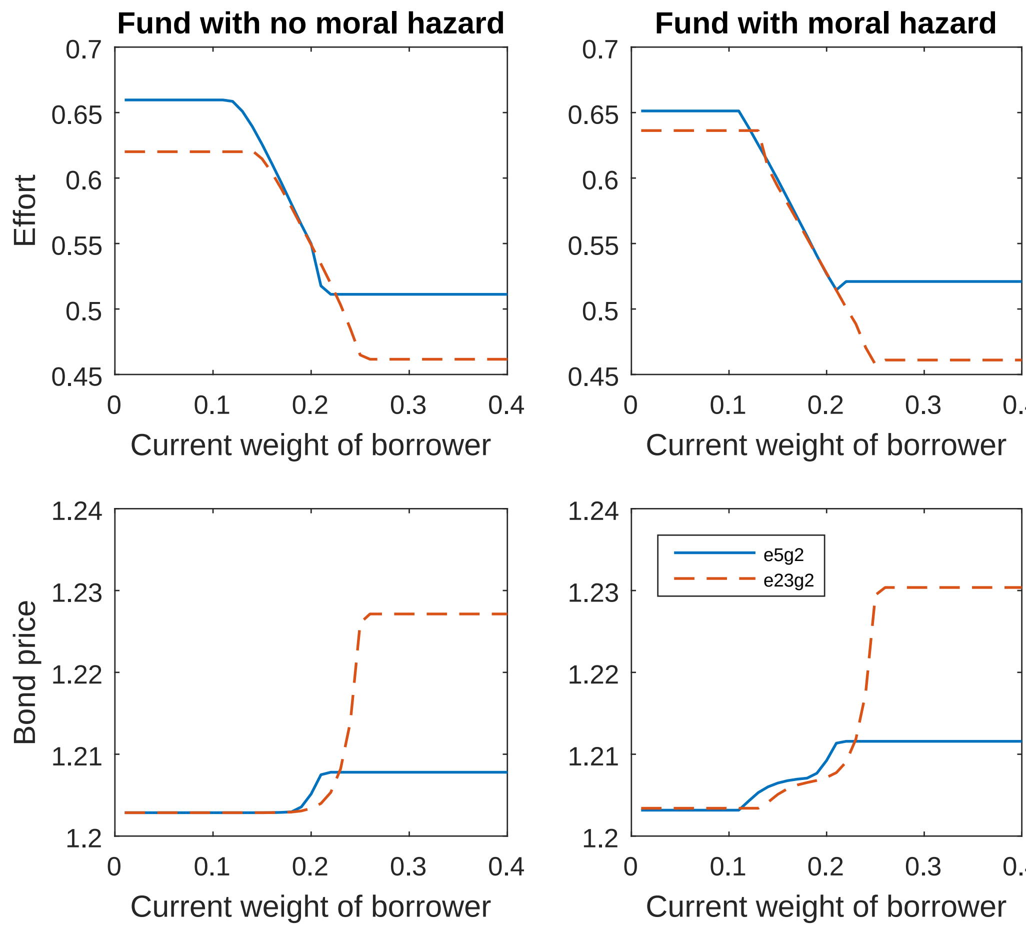

Figure 12: Effort and bond price policies: observable vs. non-observable effort

8 Simplifying the Fund contract

The *Financial Stability Fund* (Fund) described in Section 3 is complete in the sense that, subject to enforcement and incentive compatibility constraints, the contract is contingent to all shocks. However it may be difficult to implement a contract with many contingencies, even more to decentralize it. In this section we explore how the Fund contract can be simplified by making it less contingent. In other words, even if the economy is subject to many shocks, the fund only provides risk-sharing to a coarse set of shocks, i.e. fewer subsets of shocks, which we will call *regimes*. To characterize the *regime-contingent fund contract* we need to introduce some notation. Recall that $S = \{(\theta, G)\}$, is the set of finite exogenous state. Let R be the set of *regimes* and $\rho : S \to R$ be the map that assigns *all* the states in S; i.e. if $r \in R$ then $\rho^{-1}(r) \subset S$ is the set of all the states in regime r – in particular, if $r \neq r'$ then $\rho^{-1}(r) \cap \rho^{-1}(r') = \varnothing$. The transition matrix between states in S defines a transition matrix between regimes in R. The Fund contract – and, therefore, its decentalization – is only regime-contingent; for example, if there is moral hazard, the incentive compatibility constraint

(6) only depends on r^t and, therefore, effort will regime-dependent.

We assume that the *regime-contingent fund contract* is parsimonious – or *robust* – within a regime. In particular, it takes the highest outside value that the borrower can achieve within any state of a regime as the outside value of all the states in that regime; that is, the borrower's Lagrange multiplier associated with a regime is the Lagrange multiplier that guarantees that the borrower does not quit in that regime. As we will see, this has an immediate implication: within a regime – say, r – consumption is either decaying or constant if there is no moral hazard, as if it was an isolated economy with a single borrower's participation constraint; formally, $V^a(r) = \max_{s\in\rho^{-1}(r)} V^a(s)$. For example, within unconstrained states in a regime r the relative Pareto weigh, and consumption, of the borrower decreases according to $x' = \eta x$ until $V^a(r)$ prevent them from falling further. Similarly, the lender's limited enforcement constraint can not depend on s but on r; however, in our context Z is independent of both. Formally, the saddle-point Bellman equation (9) takes now the form:

$$SFV(x,r;s) = \text{SP} \min_{\left\{v_b, v_l, \widetilde{\xi}\right\}} \max_{\{c,n,e\}} \left\{ x\left[(1+v_b)\, U(c,\,n,\,e) - v_b V^a(r) - \widetilde{\xi} v'(e)\right] \right.$$

$$\left. + \left[(1+v_l)\left(\theta f(n) - G - c\right) - v_l Z\right] + \frac{1+v_l}{1+r}\mathbb{E}\left[SFV(x', r'; s') \mid r, e\right] \right\} \tag{17}$$

$$\text{where } x' = \frac{1+v_b+\varphi(G' \mid G, e)}{1+v_l}\eta x \text{ and } \varphi(G' \mid G, e) = \widetilde{\xi}\frac{\partial \pi(G'|G,e)/\partial e}{\pi(G'|G,e)}.$$

The characterization of the *Fund contract* discussed in Section 3 is slightly different. Most policies are regime-dependent – i.e. $c^b(x,r), e(x,r)$, and $\tau(x,r)$– but labour is a function of the the state, $n^b(x,s)$, since production decisions depend on productivity shocks θ. In other words, the borrower decides a s contingent labour policy, even if the Fund contract only specifies r contingent policies (we represent this difference with the notation $SFV(x, r;\, s)$, where it is assumed that $\rho(s) = r$). In parallel to the results of Section 3, these policies solve the equations:

$$u'(c^b(x,r)) = \frac{1+v_l(x,r)}{1+v_b(x,r)}\frac{1}{x}, \quad \frac{h'(1-n^b(x,s))}{u'(c^b(x,\rho(s)))} = \theta f'(n^b(x,s)) \text{ and}$$

$$\tau(x,r) = \mathbb{E}\left[\tilde{\tau}(x,s) | s \in \rho^{-1}(r)\right], \text{ where}$$

$$\tilde{\tau}(x,s) = \theta(s) f\left(n^b(x,s)\right) - c^b(x,\rho(s)) - G(s).$$

Therefore, there is a residual $\hat{\tau}(x,s|\,r) = \tilde{\tau}(x,s) - \tau(x,r)$ if $\sigma(s) = r$, and $\hat{\tau}(x,s|\,r) = 0$ if $\sigma(s) \neq r$. This residual is unaccounted by the fund contract and correspond to – possibly small – uninsured fluctuations. The assignment of these residual transfers can be done in different ways, as long as the the enforcement and the incentive constraints are regime-dependent, determining consumption smoothing and constant effort within regimes[14], while labour is state-dependent, which guarantees that constrained efficiency is preserved. In other words, as long as the commitment to pay the transfers is maintained, the borrower has no incentive to use his residual rents to change his effort or consumption and, therefore, the solution to (17) is aligned with his interests. To see this, consider the alternative assignment where the fund transfers are also s dependent – i.e. $\tilde{\tau}(x,s)$–, but enforcement

[14]Note that the fact that labour supply and effort are separable is a key element for this result.

and incentive constraints are regime-dependent, in this case there are no residual transfers and the policies are the same, so the borrower cannot improve his allocation when there are residual transfers that in expected terms must cancel out. Nevertheless, there is a loss of efficiency when regime-contingent (and not state-contingent) enforcement constraints are more stringent, as they usually are by design.

As in Section 4 the 'Simplified Fund' can be decentralized through prices that are regime-dependent, since all the constraints are regime-dependent. In fact, these prices can also provide the evaluation of the residual rents. However, whether the same solution to (17), or only an approximation, can be decentralized depends on which financial assets the borrower can use to smooth within-regimes income fluctuations. We illustrate this with two different asset structures: Arrow securities and uncontingent one period debt. In both cases the borrower has access to regime-contingent long-term assets that implement the regime-contingent part of the Fund contract. Consider first that in addition the borower can trade one-period, regime-dependent, Arrow securities; i.e. conditional on being in state r he can trade $\hat{b}(s|\,r)$ securities which guarantee zero units of consumption in state $\tilde{s}$ if $\sigma(\tilde{s}) \neq r$ and one unit of consumption in state s if $\sigma(s) = r$. Conditional on a regime, r, these securities are priced at $p(s|\,r)$, which satisfy $\sum_{s\in\sigma^{-1}(r)} p(s|\,r) = 1$. It is assumed that these securities are traded at the beginning of the contract before the state s is realized, as insurance for the initial within-regime fluctuations. In particular, in the corresponding competitive equilibrium –in regime r and state $s \in \sigma^{-1}(r)$ – the borrower solves the following problem (to simplify the exposition we eliminate Pigouvian taxes)[15]:

$$
\begin{aligned}
W^{bsu}(a, \hat{b}, r;\, s) &= \max_{\left(c,\, n,\, e,\, a(r'), \hat{b}(s'|r')\right)} \left\{ U(c,\, n,\, e) + \beta \mathbb{E}\left[W^{bsu}(a(r'), \hat{b}(s'), r';\, s') \mid r; s \right] \right\} \\
\text{s.t. } & c + q(r)\left(a'(r) - \delta a(r)\right) + \sum_{r'|r} q\left(r'|r\right) \hat{a}(r') + \sum_{r'|r} q\left(r'|r\right) \sum_{s'\in\sigma^{-1}(r')} p(s'|\,r')\hat{b}(s'|\,r') \\
& \leq \theta(s) f(n) - G(s) + (1 - \delta + \delta\kappa)\, a(r) + \hat{b}(s|\,r) \\
& \text{and, for all } r',\ a(r') \geq A_b\left(r'\right), \text{ where as before}
\end{aligned}
$$

$a\left(r'\right) = a'(r) + \hat{a}\left(r'\right)$, $\sum_{r'|r} q\left(r'|r\right) \hat{a}(r') = 0$, and $q(r) = \sum_{r'|r} q\left(r'|r\right)$; i.e. $a'(r) = \frac{\sum_{r'|r} q\left(r'|r\right) a(r')}{q(r)}$. Furthermore, $\sum_{s'\in\sigma^{-1}(r')} p(s'|\,r')\hat{b}(s'|\,r') = 0$.

The endogenous borrowing constraint $A_b(r')$ is defined as follows: let $\bar{s} \in \rho^{-1}(r)$ be the state that satisfies $V^a(\bar{s}) = \max_{s\in\rho^{-1}(r)} V^a(s) = V^a(r)$, then $W^{bsu}(A_b(r'), r';\, \bar{s}') = V^a(r')$. Note that, as before, the first order conditions for $a(r')$ satisfy:

$$
Q(r) \geq \beta \sum_{r'|r} \pi(r'|r) \frac{u'(c(r'))}{u'(c(r))},
$$

with equality if $a(r') > A_b\left(r'\right)$, for all r'. The portfolio of Arrow-insurance securities, $\{\hat{b}(s'|r')\}$ can be isolated from intertemporal regime effects which are absorbed by $a(r')$; therefore, $p(s'|\,r') = \pi(s'|r')$ and the agent fully smooth his, within the regime, residual income.

[15]We denote the value function W^{bsu} to indicate that it is the **b**orrower's value of the **s**implified fund contract when he is **u**nconstrained.

The lender (i.e. the Fund) solves a simpler – which explains the title of this subsection! – problem[16]:

$$\begin{aligned}
&W^{ls}(a,r) = \max_{(c,\,a(s'))} \left\{ c + \frac{1}{1+r}\mathbb{E}\left[W^{ls}(a(s'),r') \mid s\right] \right\} \\
&\text{s.t. } c + q(r)\left(a'(r) - \delta a(r)\right) + \sum_{r'|r} q\left(r'|r\right)\hat{a}(r') = \left(1-\delta+\delta\kappa\right)a(r) \\
&a\left(r'\right) \geq A_l\left(r'\right),
\end{aligned} \tag{18}$$

where $W^{ls}(A_r(r'), r'; s') = Z$.

Accounting for clearing market equations, it is a standard argument to show that a solution of (17) can be decentralized with the financial structure just described and, conversely, that the competitive equilibrium solutions of this economy are as efficient as the solutions to the 'Simplified Fund' contract. In particular, a 'Simplified Fund' contract has a ***primary surplus*** given by:

$$q(r)(a'(r) - \delta a(r)) - (1-\delta+\delta k)\,a(r) = \tau(x,r)$$

and a ***residual primary surplus*** given by: $-\hat{b}(s|\,r) = \hat{\tau}(x,s|\,r)$.

However, it may well be that there is no market for Arrow type securities to provide insurance within the states of a regime. In fact, this may be the case if the fluctuations within regimes are small enough that there is no much welfare loss from having a simpler financial structure to insure these fluctuations. As an example, consider the case where residual income fluctuations must be decentralized through a one-period uncontingent bond of zero net supply before trading of assets takes place, which the borrower can use to smooth intra-regime fluctuations. The relation between the bond market and the (decetralized) fund contract can take different forms represented by different modelling choices. They all have in common the following part of the the borrower's problem:

$$\begin{aligned}
&W^{bsc}(a,\tilde{b},r;\,s) = \max_{(c,\,n,\,e,\,a(r'),b')} \left\{ U(c,\,n,\,e) + \beta\mathbb{E}\left[W^{bsc}(a(r'),\tilde{b}',r';\,s') \mid r;s\right] \right\} \\
&\text{s.t. } c + q(r)\left(a'(r) - \delta a(r)\right) + \tilde{Q}(s,b'|r)\tilde{b}'(s) \leq \theta(s)f(n) - G(s) + \left(1-\delta+\delta\kappa\right)a(r) + \tilde{b}
\end{aligned}$$

.

The differences across these mixed economies are given by three interdependent elements: *i*) the bond contract and its price, represented above in the ***residual primary surplus***: $\tilde{Q}(s,b'|r)\tilde{b}'(s) - \tilde{b}$; *ii*) the endogenous – or, partially exogenous – borrowing constraint: $\tilde{b}'(s') + a(r') \geq A_b\left(r'\right)$, and *iii*) the choice of $A_b(r)$, together with $W^{bsc}(A_b(r),\tilde{b},r;\,\bar{s}) = V^a(r)$. For example, we can consider the following three different environments:

1. **Partially incomplete markets with default (PIMD).** To smooth intra-regime shocks the borrower is in a similar situation than in the incomplete markets economy (IMD) of Section 2; in particular: *i*) takes the above form, denoting the possibility that government debt liabilities are priced with a *positive spread*; *ii*) the fund contract is isolated from the bond market by having 'absolute seniority' and, therefore, it only needs to consider $a(r') \geq A_b\left(r'\right)$, with *iii*) as it has been designed in the above fund with Arrow securities (although the values would not be the same in both economies); i.e. with $W^{bsc}(A_b(r),0,r;\,\bar{s}) = V^a(r)$.

[16]Note that r denotes both 'interest rate' and 'regime', but there should not be confusion.

Although a version of this framework is implicit in several existing proposals where, on the one hand, the Fund is limited to a very limited form of insurance (e.g. against 'large economic shocks' often called a Fund for a 'rainy day'), there is a credibility problem – in particular, if the regime system is narrowly defined and the residual debt remains an important part of government's liabilities –: not having the right incentives the borrower can arrive to a situation where spreads are positive, which is usually associated with an economic downturn and then claim it is a 'large economic shocks', and, therefore, claim support from the Fund, possibly revising the contract. A very strong form of commitment is needed, from the part of the Fund, not to incur in the corresponding time-inconsistency. Therfore, it seems more reasonable that the Fund, anticipating this problem, considers other alternative designs...

2. **Partially incomplete markets with an 'exogenous'debt limit (PIML).** The Fund sets a fixed limit to the level of debt that a borrower can have to participate in a fund contract with exclusion from the Fund if such limit is violated[17]. 'Properly defined' such limit – say, $\bar{b} < 0$ – should make the debt non-defaultable, therefore: *i*) takes the form: $Q(r)b'(s) - b$; *ii*) to the endogenous borrowing constraint $a(r') \geq A_b\,(r')$ there is an additional debt constraint: $b \geq \bar{b}$ and, as in (PIMD), *iii*) $W^{bsc}(A_b(r), 0, r;\, \bar{s}) = V^a(r)$.

   Again, versions of exogenous debt limits are common, as common are the credibility problems associated with them. More specifically, the problem with the exogenous-endogenous mix is in how to 'properly define' $\bar{b} < 0$, which naturally take us to consider:

3. **Partially incomplete markets with 'endogenous'debt limits (PIME).** In this case the Fund takes into account the external debt level as part of the 'state'; in particular: $W^{bsc}(A_b(r), \tilde{b}, r;\, \bar{s}) = V^a(r)$ *i*) takes the form: $Q(r)b'(s) - b$; *ii*) the endogenous borrowing constraint is now $a(r') + b \geq A_b\,(r')$, and *iii*) $W^{bsc}(A_b(r), \hat{b}(\bar{s}), r;\, \bar{s}) = V^a(r)$, where

$$\hat{b}(\bar{s}) = \theta(s)f(n) - G(\bar{s}) - \left(q(r)\left(a'(r) - \delta a(r)\right) - \left(1 - \delta + \delta\kappa\right) a(r)\right),$$

   and, as before, $V^a(\bar{s}) = \max_{s \in \rho^{-1}(r)} V^a(s) = V^a(r)$. That is, since in the state in regime r, where the outside value is higher, $\bar{s} \in \sigma^{-1}(r)$, the borrower should be saving for other states, within r, the fund contract accounts for these savings. Of course, a stringent participation constraint may be considered to strengthen the robustness of the contract – say, $W^{bsc}(A_b(r), \hat{b}, r;\, \bar{s}) = V^a(r)$, for some $\hat{\hat{b}} \in [0, \hat{b}(\bar{s}))$ – at the cost of lowering the efficiency of the contract. In sum, in a (PIME) regime, the borrower's problem becomes:

$$W^{bsc}(a, b, r;\, s) = \max_{(c,\, n,\, e,\, a(r'), b')} \left\{U(c,\, n,\, e) + \beta \mathbb{E}\left[W^{bsc}(a(r'), b', r';\, s') \mid r; s\right]\right\}$$

$$\text{s.t. } c + q(r)\left(a'(r) - \delta a(r)\right) + Q(r)b'(s) \leq \theta(s)f(n) - G(s) + \left(1 - \delta + \delta\kappa\right) a(r) + b$$

s.t. $a(r') + b \geq A_b$

.

[17]One can also consider milder (more credible?) punishments that may also act as deterrent from violating the debt limit; e.g. a substantial reduction of future positive transfers from the fund or a temporary freeze of the fund contract (i.e. changing the default option to returning with probability λ to the Fund, instead of to the IMD market of Section 2). We do not elaborate here these small, but relevant, variations.

The lender solves the same problem (18) and the 'Simplified Fund' contract's ***primary surplus*** is equally defined:

$$q(r)(a'(r) - \delta a(r)) - (1 - \delta + \delta k)\, a(r) = \tau(x, r).$$

The ***residual primary surplus*** is: $Q(r)b'(s) - b$. Since the one period debt is traded at the within-regime riskless price $Q(r)$ the borrower smooths his (decaying) consumption as in the economy with Arrow securities, however the less efficient debt trading results in lower levels of consumption (i.e. there are more savings in this economy).

In sum, the simplified, and decetralized, fund contract provides risk-sharing across a limited number of regimes and, therefore, the contract depends on these regimes and the asset liabilities that the borrower has with the fund (a, r), except that it also account and limits the external level of debt that the borrower uses, b, to smooth intra-regime fluctuations, transforming this debt into a safe asset; i.e improving the overall performance of the economy, not only the inter-regime risk-sharing (in the PIML economy and, in particular, in the more efficient PIME economy). However, there is a tradeoff between *simplicity* for the lender (Fund) –i.e. risk-sharing across few regimes – and/or the borrower – i.e. a simple financial structure to smooth intra-regime fluctuations – and *efficiency*. How severe is this tradeoff is in terms of welfare is a quantitative question, in need of a quantitative answer.

9 Conclusions

By developing and computing a model of a *Financial Stability Fund* as a *constrained efficient mechanism* we have contributed to the existing literature on risk-sharing and sovereign debt and to the current policy debate on risk-sharing and shock-absorbing mechanisms for the Economic Monetary Union (EMU). In particular, we have quantitatively shown that the visible welfare gains of a well designed *Fund* can be substantial, even if we have calibrated the model to euro area 'stressed countries', and we have set a 'tight constraint' on risk-sharing transfers: the fund should always have non-zero expected profits from its Fund contracts. We have also shown that accounting for *moral hazard* does not substantially change the Fund allocations, whereas incentive compatibility constraints interact with limited enforcement constraints, distorting effort and making *negative spreads* more likely to emerge. In our economies, the moral hazard problem only affects the distribution of government expenditures. If, however, it were to affect productivity shocks too (e.g. through costly structural reforms) the effect may be greater. We have also shown how the fund can be simplified – in the sense of making it less contingent or relaying on a simpler financial structure – as it is has been proposed (e.g. a *Fund* to absorb 'large economic shocks'). The advantage of our framework is that is allows for a characterizations and quantitive evaluation of the tradeoff between *simplicity* and *efficiency*, providing a guide for further 'contractual engineering' work which should help its implementation.

While the *Fund* has been designed as a *risk-sharing mechanism*, we have shown it is also an effective and "robust crisis management" mechanism. Furthermore, fund contracts help to *stabilize* the economy by generating and enhancing 'counter-cyclical fiscal policies'. The *Fund* can also be used to address sovereign 'debt overhang' problems since it has high absorbing capacity; in particular, it is the self-enforcing stabilisation and default-free nature of the *Fund* what gives its credibility and its capacity to absorb large existing debts – or provide generous credit in times of crisis, – in contrast with existing debt market instruments. Existing crisis-resolution institutions – such as the ESM – are able to absorb relatively large debts, but by relying on *ex-ante* conditionality, instead of relaying on *ex-post* conditionality, as in our *constrained efficient mechanisms*, they do not exploit all the potential welfare gains of having long-term contracts[18]. Our work may be useful

[18]For example, as of May 2017, the ESM is holding 49.4% of Greece's sovereign debt (which amounts to 88.5% of

to them.. Finally, a central feature of the *Fund* is its capacity to substitute risky debt contracts with safe assets – the fund contracts – and to play a leading role in lowering the risk on other private debt contracts.

Appendix

9.1 Data sources

Table 5: Data sources and definitions

Series	Time periods	Sources	Unit
Output	1980–2015	AMECO (OVGD)[a]	1 billion 2010 constant euro
Private consumption	1980–2015	AMECO (OCPH)	1 billion 2010 constant euro
Government consump.	1980–2015	AMECO (OCTG)	1 billion 2010 constant euro
Working hours	1980–2015	AMECO (NLHT)[b]	1 million hours
Employment	1980–2015	AMECO (NETD)	1000 persons
Government debt	1980–2015	AMECO EDP[c]	end-of-year percentage of GDP
Primary surplus	1980–2015	AMECO (UBLGIE)[d]	end-of-year percentage of GDP
Bond yields	1980–2015	AMECO (ILN)[e]	percentage, nominal
Inflation rate	1980–2015	AMECO (PVGD)	percentage, GDP deflator
Debt maturity	1990–2010	OECD[f]	years
Labor share	1980–2015	AMECO[†]	percentage

a Strings in parentheses indicate AMECO labels of data series.

b PWT 8.1 values for Greece in 1980–1982.

c General government consolidated gross debt; ESA 2010 and former definition, linked series.

d AMECO linked series for 1995–2015; European Commission *General Government Data* (GDD 2002) for 1980–1995.

e A few missing values for Greece and Portugal replaced by Eurostat long-term government bond yields.

f Differing time coverage across countries; see the text for details.

† Calculated based on various series on labor compensation; see the text for details.

9.2 Solution Method

9.2.1 The Solution of the IMD

In what follows, we describe the computational algorithm to solve for the IMD model with no moral hazard.

Solving for the labor supply For given (s, b) and b', we can solve for the optimal labor from the optimality condition. If the borrower chooses not to default, the optimal labor supply n^* solves:

$$h(n) \equiv (\theta n^{\alpha} - \chi)\, n^{1-\alpha} - \vartheta(1-n)^{\sigma} = 0$$

Greece GDP) as long-term, over 30 years, unconditional debt.

where $\vartheta = (\theta\alpha)/\gamma > 0$ and $\chi = G - (1-\delta+\delta\kappa)b + q(s,b')(b'-\delta b)$. Since $h(1) = (\theta - \chi)$ and $h(0) = -\vartheta < 0$, there exists an $n^* \in (0,1)$ such that $h(n^*) = 0$ and $c^* > 0$ if and only if $\theta - \chi > 0$. It is easy to show that n^* is unique. If the borrower chooses to default, we can use the same condition with $\vartheta = \theta^p\alpha/\gamma$ and $\chi = G$.

In what follows, we denote by $N_{\text{nd}}(s,b,b')$ the optimal labor supply in the case of no default, given the current state (s,b) and the bond choice for the next period b'; and we use $N_{\text{d}}(s)$ to denote the optimal labor supply in the case of default. Here we have chosen to suppress the dependence of N_{nd} on the bond price $q(s,b')$ for two reasons: first, given any pricing function $q(\cdot)$, the specific value of the bond price is determined by (s,b'); and second, to enhance computational efficiency, we will rewrite $N_{\text{nd}}(\cdot)$ as a function of θ and χ, where χ summarizes all the dependence of N_{nd} on G, b, b', and $q(s,b')$.

Solving the Bellman Equation To find a solution to the model, we combine equations (1)-(3) as well as the pricing equation in (4) into one Bellman equation of four functions: three value functions and one pricing function. We can then use backward induction to solve the functional equation. More precisely, let $V^{bi}(s,b;k-1)$, $V^{bi}_n(s,b;k-1)$, $V^{ai}(s;k-1)$, and $q(s,b';k-1)$ denote the value and pricing functions obtained in the k'th iteration. We first solve

$$\begin{aligned} V_n^{bi}(s,b;k) = \max_{c,b'} \ & U(c, 1 - N_{\text{nd}}(s,b,b';k)) + \beta\mathbb{E}\left[V^{bi}(s',b';k-1)|s\right] \\ \text{s.t. } & c + q(s,b';k)(b'-\delta b) \leq \theta[N_{\text{nd}}(s,b,b';k)^{\alpha} - G_c - G_d + (1-\delta+\delta\kappa)b, \end{aligned} \tag{1}$$

and

$$\begin{aligned} V^{ai}(s;k) = & U(c, 1 - N_{\text{d}}(s)) + \beta\mathbb{E}\left[(1-\lambda)V^{ai}(s';k-1) + \lambda V^{bi}(s',0;k-1)|s\right] \\ & \text{s.t. } c = \theta^p[N_{\text{d}}(s)]^{\alpha} - G_c - G_d, \end{aligned} \tag{2}$$

so that

$$V^{bi}(s,b;k) = \max\{V_n^{bi}(s,b;k), V^{ai}(s;k)\}. \tag{19}$$

As explained earlier, we denote the labor supply function in the no default case by $N_{\text{nd}}(s,b,b';k)$ to make explicit the dependence of $N_{\text{nd}}(\cdot)$ on the bond pricing function $q(\cdot;k)$ in each iteration. This is a standard dynamic programming problem that delivers value and policy functions for consumption, labor and bond choices, as well as default decisions. Once we have these, we can update the pricing function via

$$q(s,b';k+1) = \mathbb{E}\left[(1 - D(s',b';k))\frac{(1-\delta) + \delta[\kappa + q(s',b(s',b';k);k)]}{1+r}|s\right], \tag{20}$$

where $D(s,b;k)$ and $b(s,b;k)$ are the default and bond holding decisions obtained in iteration k. In general, this shows that $q(\cdot;k)$ is obtained in iteration $k-1$.

To implement the backward induction algorithm, we use discrete space value function iteration. Since (θ, G_c) is discrete by assumption, we only need to discretize G_d and b. In particular, we set G_d to be equally spaced over $[-\bar{m}, \bar{m}]$ with N_d grid points, and with equal probability on each grid point for simplicity. Moreover, we discretize the bond holding space $\mathcal{B}$ with N_b grid points. We iterate on the value function and the pricing function on the discretized space $\Theta \times \mathcal{G}_c \times \mathcal{G}_d \times \mathcal{B}$ until convergece, namely, until

$$\max|V^{bi}(s,b;k) - V^{bi}(s,b;k+1)| \text{ and } \max|q(s,b';k) - q(s,b';k+1)|$$

are both smaller than some convergence criterion. Moreover, we use two parameters $\zeta_V, \zeta_q \in [0,1]$ to control the updating speed of $V^{bi}(\cdot)$ and $q(\cdot)$ as follows:

$$\begin{aligned} V^{bi}(s,b;k+1) &= \zeta_V V^{bi}(s,b;k) + (1-\zeta_V)\text{RHS of (19)}, \\ q(s,b';k+1) &= \zeta_q q(s,b';k) + (1-\zeta_q)\text{RHS of (20)}. \end{aligned}$$

Setting $\zeta_q > 0$ is useful for the convergence of $q(\cdot)$ as well.

Note that it is important to have a continuously distributed G_d to smooth off discrete changes in $D(s, b)$ and enhance the convergence properties of the model. In principle, we could keep G_d as a continuous state variable in the computation, and use the involved procedure of Chatterjee and Eyigungor (2012) to obtain the functions $D(\cdot, G_d, \cdot)$ and $b(\cdot, G_d, \cdot)$ accurately. Instead, we use a discrete approximation of G_d, which is straightforward to implement, and we find that such an approximation works good enough to improve the convergence properties of the algorithm to compute our model.

Note also that $q(s, b')$ does not depend on G_d, and this simplifies the iterations of $q(s, b'; k)$. Also, in the backward iteration, we use the fact that $q(s', b(s', b'; k); k)$ is simply the equilibrium bond price in state (s', b'), the value of which has already being computed in solving for the optimal bond choice $b(s', b'; k)$. Let $q^{\mathrm{e}}(s, b; k)$ denote the equilibrium bond price in state (s, b) under optimal bond choice $b(s, b; k)$, then (20) can be simplified into

$$q(s, b'; k+1) = \mathbb{E}\left[(1 - D(s', b'; k))\frac{(1-\delta) + \delta[\kappa + q^{\mathrm{e}}(s', b'; k)]}{1+r}|s\right].$$

The above expression implies that two equivalent ways of updating the bond prices. The first is to compute $q(\cdot; k+1)$ in the k'th iteration, after obtaining the default decision $D(\cdot; k)$. The second is to compute $q(\cdot; k+1)$ in iteration $k+1$ for each bond choice b', using default decisions obtained in the previous iteration. The latter will be useful when we implement the moral hazard case.

Improving on Efficiency In the preceeding algorithm, we do the computation of optimal labor supply $N_{\mathrm{nd}}(s, b, b')$ under no default *within* the main loop. To improve on efficiency we can use an approximation of $N_{\mathrm{nd}}(s, b, b'; k)$. As shown before, for given (s, b, b') and bond pricing function $q(s, b'; k)$, the optimal labor n^* can be written as a function of θ and χ. Since $G > 0$, $0 \leq q(s, b') \leq \bar{q} = \frac{1-\delta+\delta\kappa}{1-\delta+r}$ and $b_{\min} < 0 \leq b_{\max}$, we have $\chi_{\min} \leq \chi \leq \chi_{\max}$, where

$$\chi_{\min} = G_{\min} + \bar{q}[b_{\min} - (1+r)b_{\max}],$$
$$\chi_{\max} = G_{\max} + \bar{q}[b_{\max} - (1+r)b_{\min}].$$

Therefore, we can discretize the interval $[\chi_{\min}, \chi_{\max}]$ into a fine grid $\mathcal{X}$ with N_χ equally spaced points, and then solve for n^* over the grid $\Theta \times \mathcal{X}$ once and for all *outside* the main loop. Denote this solution by $N^*_{\mathrm{nd}}(\theta, \chi)$. To evaluate $N_{\mathrm{nd}}(s, b, b'; k)$ within the loop, we can simply interpolate N^*_{nd} for the level of χ implied by (s, b, b').

9.2.2 The Solution of the *Fund*

Using the functional forms above, the equilibrium conditions for the *Fund* can be rewritten as:

$$c(x, s) = \frac{1 + \upsilon_b(x, s)}{1 + \upsilon_l(x, s)}x,$$
$$c(x, s)\,\gamma\,(1 - n(x, s))^{-\sigma} = \theta\alpha n(x, s)^{\alpha-1},$$
$$x(s') = \frac{1 + \upsilon_b(x, s) + \varphi(x, s')}{1 + \upsilon_l(x, s)}\eta x,$$

where $\varphi(x,s')$ is given by

$$\varphi(x,s') = \tilde{\xi}(s)\frac{\rho\exp(-\rho e)\left[\pi^g(G'|G) - \pi^b(G'|G)\right]}{\pi^G(G'|G,e)}$$

and $\tilde{\xi}(s)$ is the Lagrange multiplier of the incentive constraint in the normalized problem; i.e. $\tilde{\xi}(s) = \frac{\xi(s)}{\mu_b(s)}$. Furthermore,

$$2\omega e = \beta\rho\exp(-\rho e)\sum_{G',\theta'}\left[\pi^g(G'|G) - \pi^b(G'|G)\right]\pi^\theta(\theta'|\theta)V^{bf}(x',s'),$$

$$\begin{aligned} 0 \quad = \quad & \rho\exp(-\rho e)\sum_{G',\theta'}\left[\pi^g(G'|G) - \pi^b(G'|G)\right]\pi^\theta(\theta'|\theta)\left[\frac{1+\upsilon_l(x,s)}{(1+r)x}V^{lf}\left(x\left(s'\right),s'\right) - \tilde{\xi}(x,s)\beta\rho V^{bf}(x(s'),s')\right] \\ & -\tilde{\xi}(x,s)2\omega \end{aligned}$$

$$V^{bf}(x,s) = \log(c(x,s)) + \frac{\gamma(1-n(x,s))^{1-\sigma}}{1-\sigma} - \omega e^2 + \beta\sum_{s'\in S}\pi^G(G'|G,e)\pi^\theta(\theta'|\theta)V^{bf}(x(s'),s').$$

$$V^{lf}(x,s) = \theta n(x,s)^\alpha - c(x,s) - G + \frac{1}{1+r}\sum_{s'\in S}\pi^G(G'|G,e)\pi^\theta(\theta'|\theta)V^{lf}(x(s'),s').$$

$$V^{af}(s) = \max_n\left\{\begin{array}{c} \log(\theta n^\alpha - (1-\phi)G) + \frac{\gamma(1-n)^{1-\sigma}}{1-\sigma} - \omega e^2 + \\ +\beta\sum_{s'\in S}\pi^G(G'|G,e)\pi^\theta(\theta'|\theta)V^{af}(s') \end{array}\right\}$$

The solution to this system of equations is found numerically using a policy iteration algorithm. More precisely, we discretize the relative pareto weight for the borrower x. For each grid point, we can calculate the value of autarky by solving for the optimal labor in autarky first and calculating $V^{af}(s)$ from the previous equation.

We then define the region of pareto weights between which none of the participation constraints are binding. In that region, for each shock $s = (\theta, G)$, the solution is characterized by the first full commitment solution with unobservable effort but no participation constraints:

$$\begin{aligned} c(x,s) &= x \\ c\left(x,s\right)\gamma\left(1-n\left(x,s\right)\right)^{-\sigma} &= \theta\alpha n\left(x,s\right)^{\alpha-1}, \\ x\left(s'\right) &= \left[1+\widetilde{\xi}\frac{\partial\pi(G'|G,e)/\partial e}{\pi(G'|G,e)}\right]\eta x, \end{aligned}$$

$$\begin{aligned} 0 = & \sum_{s'|s}\pi^\theta(\theta'|\theta)\rho\exp(-\rho e)\left(\pi^g(G'|G) - \pi^b(G'|G)\right)\frac{1}{1+r}\frac{1}{x}V^{lf}(x',s') \\ & +\widetilde{\xi}(x,s)\left[-\sum_{s'|s}\pi^\theta(\theta'|\theta)\rho\exp(-\rho e)\left(\pi^g(G'|G) - \pi^b(G'|G)\right)\rho\beta V^{bf}(x',s') - 2\omega\right] \end{aligned}$$

$$2\omega e = \beta\rho\exp(-\rho e)\sum_{G',\theta'}\left[\pi^g(G'|G) - \pi^b(G'|G)\right]\pi^\theta(\theta'|\theta)V^{bf}(x',s'),$$

where

$$V^{bf}(x,s) = \log(c(x,s)) + \frac{\gamma(1-n(x,s))^{1-\sigma}}{1-\sigma} - \omega e^2 + \beta\sum_{s'\in S}\pi^G(G'|G,e)\pi^\theta(\theta'|\theta)V^{bf}(x(s'),s').$$

$$V^{lf}(x,s) = \theta n(x,s)^\alpha - c(x,s) - G + \frac{1}{1+r}\sum_{s'\in S}\pi^G(G'|G,e)\pi^\theta(\theta'|\theta)V^{lf}(x(s'),s').$$

$$V^{af}(s) = \max_n \left\{ \begin{array}{c} \log(\theta n^\alpha - (1-\phi)G) + \frac{\gamma(1-n)^{1-\sigma}}{1-\sigma} - \omega e^2 + \\ +\beta\sum_{s'\in S}\pi^G(G'|G,e)\pi^\theta(\theta'|\theta)V^{af}(s') \end{array} \right\}$$

To find the region for which the participation constraint binds for the borrower, for each shock $s = (\theta, G)$, we find $c(x_b, s) = x_b$ such that $V^{bf}(x_b, s) = V^{af}(s)$. For the decentralization, using one-period Arrow securities, the bond price simplifies to:

$$\begin{aligned} q(s'|s) &= \max\left\{\beta\pi(s'|s)\frac{u'(c(x',s'))}{u'(c(x,s))}, \left(\frac{1}{1+r}\right)\pi(s'|s)\right\} \\ &= \pi(s'|s)\max\left\{\beta\frac{c(x,s)}{c(x',s')}, \left(\frac{1}{1+r}\right)\right\} \end{aligned}$$

The price of a one period bond is then equal to:

$$q^f(s) = \sum_{s'\in S} q(s'|s)$$

which in turn implies a risk free rate of $r^f(s) = \frac{1\delta+\delta\kappa}{q^f(s)}$. Finally, we can recover the asset holdings numerically by iterating to find the asset holding function that satisfies:

$$\begin{aligned} a_b'(x,s) &= \sum_{\theta'\in S} q(s')\, a_b(x',s') + c(x,s) - \theta f(n(x,s)) + G \\ a_l(x,s) &= -a_b(x,s) \end{aligned}$$

Moreover, we define the repayment as:

$$a_b(x',s') - \sum_{s'\in S} q(s'|s)\, a_b(x',s')$$

9.3 Further notes on the calibration procedure

On the transition matrix of the G shock. Note that this specification of the transition matrix is motivated by the one-period-crash Markov chain of Rietz (1988):

$$\pi^R = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \eta & \phi & 1-\phi-\eta \\ \eta & 1-\phi-\eta & \phi \end{bmatrix},$$

where the first state is labeled as the "crash" or "crisis" state, and the associated stationary distribution is

$$\mu^R = \begin{bmatrix} \frac{\eta}{1+\eta} & \frac{1}{2(1+\eta)} & \frac{1}{2(1+\eta)} \end{bmatrix}.$$

References

Alvarez, Fernando and Urban J. Jermann, "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica*, 2000, *68* (4), 775–797.

Arellano, Cristina, "Default Risk and Income Fluctuations in Emerging Economies," *American Economic Review*, June 2008, *98* (3), 690–712.

Beraja, Martin, "A Semi-Structural Methodology for Policy Counterfactuals and an Application to Fiscal Union," Working Paper, University of Chicago 2016.

Chatterjee, Satyajit and Burcu Eyigungor, "Maturity, Indebtedness, and Default Risk," *American Economic Review*, 2012, *102* (6), 2674–99.

Dovis, Alessandro, "Efficient Sovereign Default," *Pennsylvania State University*, 2016.

Eaton, Jonathan and Mark Gersovitz, "Debt with Potential Repudiation: Theoretical and Empirical Analysis," *Review of Economic Studies*, April 1981, *48* (2), 289–309.

Furceri, Davide and Aleksandra Zdzienicka, "The Euro Area Crisis: Need for a Supranational Fiscal Risk Sharing Mechanism?," *Open Economies Review*, 2015, *26* (4), 683–710.

Hamilton, James D., "Analysis of Time Series Subject to Changes in Regime," *Journal of Econometrics*, June 1990, *45* (1-2), 39–70.

_ , "State-Space Models," in Robert F. Engle and Daniel L. McFadden, eds., *Handbook of Econometrics*, Vol. 4, Elsevier, 1994, chapter 50, pp. 3039–3080.

Krueger, Dirk, Hanno Lustig, and Fabrizio Perri, "Evaluating Asset Pricing Models with Limited Commitment Using Household Consumption Data," *Journal of the European Economic Association*, April / May 2008, *6* (2-3), 715–726.

Marcet, Albert and Ramon Marimon, "Recursive Contracts," *European University Institute*, March 2017.

Mele, Antonio, "Repeated Moral Hazard and Recursive Lagrangeans," *Journal of Economic Dynamics and Control*, May 2014, *42*, 69–85.

Tirole, Jean, "Country Solidarity in Sovereign Crises," *American Economic Review*, August 2015, *105* (8), 2333–63.